MATH 8440

Problem Set 4

Due Monday, November 9

Instructions. Turn in at least 30 points' worth of problems.

Problem 1. *primes outside a subgroup* (7 points) Let H be a proper subgroup of $(\mathbf{Z}/m\mathbf{Z})^\times$. Adapt Euclid's proof to show that there are infinitely many primes p for which $p \bmod m \notin H$.

Problem 2. (7 points) Show that for complex z with $|z| < 1$,

$$\sum_{\substack{n\geq 1 \\ p|n \Rightarrow p\equiv 3 \pmod 4}} \mu(n)\frac{z^n}{1-z^{2n}} = \sum_{\substack{m\geq 1 \\ p|m \Rightarrow p\equiv 1 \pmod 4}} z^m.$$

Suppose that there are only finitely many primes $p \equiv 3 \pmod 4$. Setting $z = iy$ and letting y tend to 1 from below, show that the left-hand side of this identity tends to a limit while the right-hand side "blows up" (has absolute value tending to infinity).

This proof is due to J. J. Sylvester.

Problem 3. (5 points) Let G be an abelian group, written additively. Suppose $f, g : \mathbf{Z}^+ \to G$ are such that $g(n) = \sum_{d|n} f(d)$ for all n. Prove that $f(n) = \sum_{d|n} g(d)\mu(n/d)$ for all n.

Problem 4. (3 points) Let m be an integer, $m \geq 3$. Prove that $\Phi_m(x)$ is self-reciprocal (palindromic).

Problem 5. (7 points) Let R be a commutative ring (with unit). Prove that if $f \in R[x]$ is self-reciprocal with even degree d, then $f(x) = x^{d/2}g(x + x^{-1})$ for a uniquely determined $g(x) \in R[x]$. (Here the identity is interpreted in $R[x, x^{-1}]$.) Furthermore, $\deg g = d/2$.

Problem 6. (3 points) Let G be a finite abelian group of size n. Order the elements of G as $g_1, \ldots, g_n$ and the characters of G as $\chi_1, \ldots, \chi_n$, and let M be the $n \times n$ matrix whose ith row, jth column entry is $\chi_i(g_j)$. Compute MM^*, where M^* is the conjugate transpose of M.

Problem 7. *a characterization of Dirichlet characters* (3 points) Let $m \in \mathbf{Z}^+$. Suppose $\chi : \mathbf{Z} \to \mathbf{C}$ satisfies

(a) χ is periodic mod m: $\chi(a) = \chi(a + m)$ for all $a \in \mathbf{Z}$,

(b) χ is completely multiplicative: $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in \mathbf{Z}$,

(c) $\chi(a) = 0 \iff \gcd(a, m) > 1$.

Prove that χ is a Dirichlet character modulo m.

Problem 8. (10 points) Let χ be a nontrivial Dirichlet character mod m. Define $S(t) := \sum_{n\leq t} \chi(n)$. Below, s is always a positive real number.

(a) Show that $\sum_{n\leq x} \chi(n)n^{-s} = S(x)x^{-s} + s\int_1^x S(t)t^{-s-1}\,\mathrm{d}t$ (for all $x \geq 1$). Deduce that $\sum_{n=1}^\infty \chi(n)n^{-s}$ converges.

(b) Show that $\sum_{n>x} \chi(n)n^{-s} = s\int_x^\infty (S(t) - S(x))t^{-s-1}\,\mathrm{d}t$ (for all $x \geq 1$). Deduce that

$$\sum_{n\leq x} \chi(n)n^{-s} = L(s, \chi) + E, \quad \text{where} \quad |E| \leq \phi(m)x^{-s}.$$

(c) Show that $\sum_{n\geq 1} \chi(n)\log n/n$ converges.

Hint. The sum of $\chi(n)$ over any block of m consecutive integers vanishes.

Problem 9. *A congruence form of Goldbach's conjecture* (5 points) Let n be an even integer. Show that for every positive integer m, there is a pair of primes p_1, p_2 with $p_1 + p_2 \equiv n \pmod{m}$.

Problem 10. (7 points) Use Dirichlet's theorem to prove that $2x+5y+7$ and $3x-2y+1$ are simultaneously prime for infinitely many pairs of integers x, y.

Problem 11. *arbitrarily long two-sided prime gaps* (5 points) Use Dirichlet's theorem to prove that for every positive integer M, there are infinitely many primes p where all of $p \pm 1$, $p \pm 2$, ..., $p \pm M$ are composite.