THE DISTRIBUTION OF INTERMEDIATE PRIME FACTORS

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Abstract. Let $P^{(\frac{1}{2})}(n)$ denote the middle prime factor of n (taking into account multiplicity). More generally, one can consider, for any $\alpha \in (0,1)$, the α-positioned prime factor of n, $P^{(\alpha)}(n)$. It has previously been shown that $\log\log P^{(\alpha)}(n)$ has normal order $\alpha \log\log x$, and its values follow a Gaussian distribution around this value. We extend this work by obtaining an asymptotic formula for the count of $n \le x$ for which $P^{(\alpha)}(n) = p$, for primes p in a wide range up to x. We give several applications of these results, including an exploration of the geometric mean of the middle prime factors, for which we find that $\frac{1}{x}\sum_{1<n\le x} \log P^{(\frac{1}{2})}(n) \sim A(\log x)^{\varphi-1}$, where φ is the golden ratio, and A is an explicit constant. Along the way, we obtain an extension of Lichtman's recent work on the "dissected" Mertens' theorem sums $\sum_{P^+(n)\le y,\ \Omega(n)=k} \frac{1}{n}$ for large values of k.

1. Introduction

Starting with [10] various papers have considered the distributional properties of the "middle prime factor" of an integer. Suppose the prime factorization of an integer $n > 1$ is written as

$$n = q_1^{a_1} q_2^{a_2} \cdots q_{\omega(n)}^{a_{\omega(n)}} = p_1 p_2 \cdots p_{\Omega(n)},$$

with $q_1 < q_2 < \cdots < q_{\omega(n)}$ and $p_1 \le p_2 \le \cdots \le p_{\Omega(n)}$. Call $Q^{(\frac{1}{2})}(n) := q_{\lceil \omega(n)/2 \rceil}$ the middle prime factor of n without considering multiplicity and $P^{(\frac{1}{2})}(n) := p_{\lceil \Omega(n)/2\rceil}$ the middle prime factor of n considering multiplicity.

The study of the middle prime factor without considering multiplicity was first taken up in [10], where the authors obtained an asymptotic for the reciprocal sum of the middle prime factors of n up to x,

$$\sum_{1<n\le x} \frac{1}{Q^{(\frac{1}{2})}(n)} = \frac{x}{\log x} \exp\left((1+o(1))\sqrt{2\log_2 x \log_3 x} \right).$$

Here and throughout this paper we write $\log_k x$ to denote the k-fold iterated natural logarithm. The above asymptotic was significantly sharpened by Ouellet in [23]; that paper also considered the problem generalized to the α-positioned prime factor $Q^{(\alpha)}(n) := q_{\lceil \alpha\omega(n)\rceil}$. In the remainder, we similarly define[1] $P^{(\alpha)}(n) := p_{\lceil \alpha\Omega(n)\rceil}$. (For convenience, we henceforth

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[1]As in our prior paper [21], we use a slightly different definition of the α-positioned prime factor than that given before by De Koninck, Doyon, and Ouellet. They take the prime factor in position $\max\{1, \lfloor \alpha(\Omega(n)+1)\rfloor\}$ rather than position $\lceil \alpha\Omega(n)\rceil$ as we do here. These two definitions often give the same position and never differ by more than 1, and the results quoted here from other papers go through with either definition.

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define $P^{(\alpha)}(1) := 1$.) The asymptotic formula for the reciprocal sum of the middle prime factors considering multiplicity [12] is surprisingly different,

$$\sum_{n\le x} \frac{1}{P^{(\frac{1}{2})}(n)} = c\frac{x}{\sqrt{\log x}}\left(1+O\left(\frac{1}{\log_2 x}\right)\right)$$

for an explicit constant $c = \dfrac{9}{4\sqrt{\pi}}\displaystyle\sum_p \frac{1}{p^2-2p}\prod_{3\le q<p}\frac{1-\frac{1}{2q}}{1-\frac{2}{q}}\prod_q \frac{(1-\frac{1}{q})^{\frac{1}{2}}}{1-\frac{1}{2q}}$.

Despite their different reciprocal sums, both $\log_2 Q^{(\alpha)}(n)$ and $\log_2 P^{(\alpha)}(n)$ have normal order $\alpha \log_2 x$ (see [9]) and both of these values are distributed according to the Gaussian law. In particular [8] shows for fixed $\epsilon \in (0, \frac{1}{8})$, $\alpha \in (0,1)$ and $|t| \ll (\log_2 x)^\epsilon$ that

$$\frac{1}{x}\#\left\{n\le x : \frac{\log_2 P^{(\alpha)}(n) - \alpha\log_2 x}{\sqrt{\log_2 x}} < t\right\} = \Phi\left(\frac{t}{\sqrt{\alpha(1-\alpha)}}\right) + O_\alpha\left(\frac{1}{\sqrt{\log_3 x}}\right) \tag{1}$$

where $\Phi(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\tau} e^{-v^2/2}dv$ is the Gaussian probability distribution function.

Recently in [21], the authors of this paper studied the distribution of the values of $P^{(\alpha)}(n)$ in a variety of settings. In particular this α-positioned prime factor obeys Benford's leading digit law, and is equidistributed in coprime residue classes modulo q where q can be nearly as large as, but not significantly larger than, $(\log x)^{c(\alpha)}$, for $c(\alpha) := 1 - 2^{-\alpha/(1-\alpha)}$.

2. Results

In this paper we investigate the distributional properties of the function $P^{(\alpha)}(n)$ in much greater detail. In particular, we determine, for a wide range of prime numbers p, an asymptotic for the number of integers up to x that have p as the middle (or α-positioned) prime factor. We define

$$M_p^{(\alpha)}(x) := \#\{n \le x : P^{(\alpha)}(n) = p\}$$

and set

$$\beta := \frac{\log_2 p}{\log_2 x}$$

(so that $\log p = (\log x)^\beta$). Throughout the remainder of the paper, p and β will be related by this expression. We first consider the middle prime factor (the case when $\alpha = \frac{1}{2}$).

Theorem 2.1. *Let $\epsilon > 0$ and suppose that p is a prime number, $p \to \infty$. Then if either $\beta < \frac{1}{5} - \epsilon$ or $\frac{1}{5} + \epsilon < \beta < 1 - \epsilon$ we have*

$$M_p^{(\frac{1}{2})}(x) = \begin{cases} \left(1 + O_\epsilon\left(\sqrt{\dfrac{\log_3 x}{\log_2 x}}\right)\right) C_\beta \dfrac{x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}}\sqrt{\log_2 x}} & \text{if } \frac{1}{5}+\epsilon < \beta < 1-\epsilon, \\ \left(1 + O_\epsilon\left(\sqrt{\dfrac{\log_3 x}{\log_2 x}} + \dfrac{(\log_2 p)^{-1/2}}{(\log p)^{\epsilon^2}}\right)\right) C \dfrac{x}{p(\log x)^{\frac{1}{2}-\frac{3}{2}\beta}} & \text{if } 0 < \beta < \frac{1}{5}-\epsilon, \end{cases} \tag{2}$$

where

$$C_\beta := \frac{\exp\left(\frac{\gamma(1-2\beta)}{\sqrt{\beta(1-\beta)}}\right)}{\Gamma\left(1+\sqrt{\frac{\beta}{1-\beta}}\right)} \frac{\sqrt{\beta}+\sqrt{1-\beta}}{2\sqrt{\pi}\beta^{1/4}(1-\beta)^{3/4}} \prod_{q \text{ prime}} \left(1-\frac{1}{q}\right)^{\sqrt{\frac{1-\beta}{\beta}}} \left(1-\frac{\sqrt{\frac{1-\beta}{\beta}}}{q}\right)^{-1} \tag{3}$$

$$C := \frac{3e^{\frac{3\gamma}{2}}}{4\sqrt{\pi}} \prod_{q>2 \text{ prime}} \left(1+\frac{1}{q(q-2)}\right) = 1.523555\ldots.$$

While the expression for the constant C_β above depends on β in a complicated way, we observe that $C_{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}$.

The difference in the asymptotic formulae for $M_p^{(\frac{1}{2})}(x)$ in the two ranges of β considered above is largely a manifestation of the difference in the asymptotic behavior of the sum of reciprocals of p-smooth numbers having k prime divisors, in the two cases when $k/\log_2 p$ is less than or greater than 2 (and bounded away from 2). For $k \le (2-\epsilon)\log_2 p$, work of Lichtman [20] (Theorem 5.1) provides the desired information, while in Theorem 5.3, we extend Lichtman's result to the case $k \ge (2+\epsilon)\log_2 p$.

We can similarly obtain a version of this theorem that holds for general α, however we need to introduce some additional notation before we can state this theorem.

First define the constants (depending on α and β)

$$\chi := \frac{(1-\alpha)\beta}{(1-\beta)\alpha} \qquad \text{and} \qquad \nu := 2^{-\frac{1}{1-\alpha}},$$

expressions which appear frequently in the statements below. We also define $\rho_{\chi,\alpha}$ and $\rho_{\nu,\alpha}$,

$$\rho_{c,\alpha} := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c^{\{n(1-\alpha)\}}$$

where $\{\theta\} = \theta - \lfloor\theta\rfloor$ denotes the fractional part of a real number, and c will be taken to be either χ or ν. Note that $\rho_{\chi,\alpha} = 1$ whenever $\alpha = \beta$, since in this case $\chi = 1$.

The constant $\rho_{c,\alpha}$ is the long term average of c raised to the fractional part of the integer multiples of $1-\alpha$. As such, the behavior of these sums depends on whether α is rational or irrational. If $1-\alpha = \frac{a}{b}$ is rational with $\gcd(a,b) = 1$, we find that

$$\rho_{c,\alpha} = \frac{1}{b}\sum_{i=0}^{b-1} c^{i/b} = \begin{cases} 1 & \text{if } c = 1 \\ \dfrac{c-1}{b(c^{1/b}-1)} & \text{if } c \ne 1. \end{cases}$$

On the other hand, if $\alpha \ne \beta$ is irrational, it follows from the equidistribution theorem (see the proof of Lemma 4.7 below for more explicit statements) that

$$\rho_{c,\alpha} = \int_0^1 c^t dt = \frac{c-1}{\log c}.$$

Finally, we define the value

$$\beta_\alpha := \frac{1}{1+2^{\frac{1}{1-\alpha}}(\alpha^{-1}-1)} = \frac{1}{1+\nu^{-1}(\alpha^{-1}-1)}.$$

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As we will see below, for fixed α, the behaviour of $M_p^{(\alpha)}(x)$ has a phase transition for β on either side of this value, analogous to the transition at $\beta = \frac{1}{5}$ when $\alpha = \frac{1}{2}$. Note that $\beta_\alpha < \alpha$ for all $\alpha \in (0,1)$.

We can now state the following generalization of Theorem 2.1.

Theorem 2.2. *Fix $\epsilon > 0$ and $\alpha \in (0,1)$. Suppose that $\beta \in (0,1)$ satisfies either $0 < \beta < \beta_\alpha - \epsilon$ or $\beta_\alpha + \epsilon < \beta < 1-\epsilon$ and that α is rational. Then we have, as $x, p \to \infty$,*

$$M_p^{(\alpha)}(x) = \begin{cases} \left(1 + O_{\alpha,\epsilon}\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right) \dfrac{C_{\beta,\alpha} x}{p(\log x)^{1-\left(\frac{\beta}{\alpha}\right)^\alpha \left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha}} \sqrt{\log_2 x}} & \text{if } \beta_\alpha + \epsilon < \beta < 1-\epsilon, \\ \left(1 + O_{\alpha,\epsilon}\left(\left(\frac{\log_3 x}{\log_2 x}\right)^{\frac{1}{4}} + \frac{(\log_2 p)^{-\frac{1}{2}}}{(\log p)^{\epsilon^2}}\right)\right) \dfrac{C_\alpha x}{p(\log x)^{1-2\beta-2\nu(1-\beta)}} & \text{if } 0 < \beta < \beta_\alpha - \epsilon, \end{cases}$$

where

$$C_{\beta,\alpha} := \frac{e^{\gamma(\chi^{\alpha-1} - \chi^\alpha)}}{\Gamma(1+\chi^\alpha)} \frac{\chi^{\frac{3}{2}\alpha - 1} \rho_{\chi,\alpha}}{\sqrt{2\pi\alpha(1-\beta)}} \prod_{q \text{ prime}} \left(1 - \frac{1}{q}\right)^{\chi^{\alpha-1}} \left(1 - \frac{\chi^{\alpha-1}}{q}\right)^{-1},$$

$$C_\alpha := \frac{\nu^\alpha \rho_{\nu,\alpha} e^{\gamma(2-\nu^\alpha)}}{2(1-\alpha)\Gamma(1+\nu^\alpha)} \prod_{q>2 \text{ prime}} \left(1 + \frac{1}{q(q-2)}\right).$$

The dependence of the implied constants on α comes from the distance of α from 0 and 1, as well as the size of the denominator of α.

The same asymptotic equalities hold true (without any effective error term) for each fixed irrational α. For almost all α in this range, they hold with an explicit multiplicative error of

$$1 + O_{\alpha,\epsilon}\left(\frac{(\log_3 x)^{3/4} (\log_4 x)^{1+\epsilon}}{(\log_2 x)^{1/4}} + \frac{1}{(\log p)^{\epsilon_0} (\log_2 p)^{1/2}}\right).$$

Finally, for any parameter $\mathcal{E} = o(1)$ as $x \to \infty$, we have, uniformly in $\beta \in (\epsilon, 1-\epsilon)$ and $\alpha \in (\beta - \mathcal{E}, \beta + \mathcal{E})$,

$$M_p^{(\alpha)}(x) = \left(1 + O\left(\mathcal{E} + \sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right) \frac{C_{\beta,\beta} x}{p(\log x)^{1-\left(\frac{\beta}{\alpha}\right)^\alpha \left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha}} \sqrt{\log_2 x}}.$$

Here the implied constant is independent of α.

The obstruction to uniformity for irrational α comes from known bounds on the discrepancy of the Kronecker sequence $\{(1-\alpha)n\}_{n=1}^N$, see the proof of Lemma 4.7. However, adapting some of the arguments for the above result, we can also give the following bound on $M_p^{(\alpha)}(x)$, which is completely uniform in all $\alpha \in (0,1)$ and in β away from 0 and 1.

Theorem 2.3. *Fix $\epsilon \in (0, 1/2)$. We have uniformly for $\alpha \in (0,1)$ and $\beta \in (\epsilon, 1-\epsilon)$,*

$$M_p^{(\alpha)}(x) \ll \frac{x}{p\sqrt{\log_2 x}}.$$

It is worth noting that the above bound is best possible in its range of uniformity, since equality is attained in the case $\alpha = \beta$ itself (as seen from Theorem 2.2). Moreover, as the proof will show, the bound in Theorem 2.3 can be improved to a power saving in $\log x$, either if α is close to 0 or 1, or if β lies in $(\delta, \beta_\alpha + \delta)$ for any δ fixed small enough in terms of ϵ.

Because of the presence and behavior of the $\rho_{c,\alpha}$ terms, the constants C_α and $C_{\beta,\alpha}$ above (the latter being viewed as a function of α for fixed β) have the property of being continuous at every irrational, but discontinuous at every rational value of α, except at $\alpha = \beta$ (if β is rational). At this value, when $\alpha = \beta$, the constant $C_{\alpha,\alpha}$ above simplifies to

$$C_{\alpha,\alpha} = \frac{1}{\sqrt{2\pi\alpha(1-\alpha)}}$$

whether α is rational or not. Using this, the normal distribution of the α-positioned prime factor (1) follows as a corollary. In fact we can improve the error term in that expression to the following.

Corollary 2.4. *Fix $\alpha \in (0,1)$. We have, uniformly for all real t,*

$$\frac{1}{x}\#\left\{n \le x : \frac{\log_2 P^{(\alpha)}(n) - \alpha \log_2 x}{\sqrt{\log_2 x}} < t\right\} = \Phi\left(\frac{t}{\sqrt{\alpha(1-\alpha)}}\right) + O_\alpha\left(\frac{(\log_3 x)^{3/2}}{\sqrt{\log_2 x}}\right). \quad (4)$$

The proof proceeds by partial summation over primes of $M_p^{(\alpha)}(x)$, combined with the methods used in the proof of Theorems 2.5 and 2.6 below.

We can similarly derive as a corollary a description of the distribution of the relative position of a fixed prime p among the prime divisors of integers divisible by p. For an integer n that is divisible by p, we write $n = p_1 \cdots p_{\Omega(n)}$, where

$$p_1 \le p_2 \le \cdots \le p_{k-1} \le p = p_k < p_{k+1} \le \cdots \le p_{\Omega(n)} \quad (5)$$

so that p is the k-th smallest prime factor of n (and if n is divisible by p^2, we take the largest index corresponding to a factor of p). For such an n we then denote by $R_p(n) = \frac{k}{\Omega(n)}$ the relative position of p among the prime factors of n. It isn't hard to show that the normal order of $R_p(n)$ is $\beta = \frac{\log_2 p}{\log_2 x}$; we show that it is in fact normally distributed around this value.

Theorem 2.5. *Fix $\epsilon > 0$. We have, for $x \ge 10$*

$$\frac{1}{x/p}\#\left\{n \le x : p|n, \frac{R_p(n) - \beta}{(\log_2 x)^{-1/2}} < t\right\} = \Phi\left(\frac{t}{\sqrt{\beta(1-\beta)}}\right) + O_\epsilon\left(\frac{1}{(\log_2 x)^{1/3}}\right), \quad (6)$$

uniformly in $\beta \in (\epsilon, 1-\epsilon)$ and all real t.

We conclude with one more application of Theorem 2.1. If we denote by $P_k(n)$ the k-th *largest* prime factor of n, Dickman [11], de Bruijn [5, Eq. 5.1] and later Knuth and Trabb Pardo [19] investigate the average value of $\log P_k(n)$ and find, for each fixed k, that

$$\frac{1}{x}\sum_{n \le x} \log P_k(n) = (D_k + o(1)) \log x$$

where the D_k are constants and in particular $D_1 = 0.624329\ldots$ is the Golomb-Dickman constant.[2]

This result can be interpreted as saying that "on average" the largest prime divisor of an integer n has just under 5/8 as many digits as n, and for any fixed k the k-th largest prime

[2] In fact, $\frac{1}{x}\sum_{n \le x} \log P_1(n) = D_1 \log x + D_1(1-\gamma) + O\left(\exp(-(\log x)^{3/8-\epsilon})\right)$, see [25, Exercise 290].

factor of n has, on average, a fixed, positive proportion of the number of digits that n has. Tenenbaum [24, Corollary 4] considers this same average for the least prime factor $P^-(n)$ and shows (for a complicated but explicit constant A^-) that

$$\frac{1}{x}\sum_{1<n\leq x}\log P^-(n) = e^{-\gamma}\log\log x + A^- + O\left(\exp(-(\log x)^{3/8-\epsilon})\right).$$

We investigate the same problem for the middle (or α-positioned) prime factor of n.

Theorem 2.6. *Let $\varphi = \frac{1+\sqrt{5}}{2}$ be the golden ratio, and $\varphi' = \frac{1}{\varphi} = \varphi - 1 = \frac{\sqrt{5}-1}{2} = 0.6180\ldots$ its reciprocal. The average value of the logarithm of the middle prime factor of the integers up to x satisfies*

$$\frac{1}{x}\sum_{n\leq x}\log P^{(\frac{1}{2})}(n) = A(\log x)^{\varphi'}\left(1 + O\left(\frac{(\log_3 x)^{3/2}}{\sqrt{\log_2 x}}\right)\right) \tag{7}$$

where

$$A := \frac{e^{-\gamma}}{\varphi!}\frac{\varphi+1}{\sqrt{5}}\prod_p\left(1-\frac{1}{p}\right)^{\varphi'}\left(1-\frac{\varphi'}{p}\right)^{-1} = 1.313314\ldots$$

and $\varphi! := \Gamma(\varphi+1) = \Gamma(\varphi-1) = \Gamma(\varphi')$.

Note that φ is one of the solutions to the equation $\Gamma(x+1) = \Gamma(x-1)$.

Remark 2.7. One can similarly generalize this to other values of α. Using Theorem 2.2 one can similarly derive that for any fixed $0 < \alpha < 1$ we have $\frac{1}{x}\sum_{n\leq x}\log P^{(\alpha)}(n) \sim A_\alpha(\log x)^{B_\alpha}$, where $B_\alpha = \max\limits_{0<\beta<1}\{\beta + \left(\frac{\beta}{\alpha}\right)^\alpha\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha} - 1\}$.

2.1. Background on the distribution of the j-th smallest and largest prime factor.

Other authors have investigated the distribution of intermediate prime factors from a slightly different perspective than that considered here, namely by considering directly the j-th smallest prime factor of an integer. We include here some of the notable results from this perspective for comparison.

Most directly comparable to our work is the work of Galambos [14], extending work of Erdős [13], which shows the following. As before, let $p_j = p_j(n)$ denote[3] the j-th smallest prime factor of n counted with multiplicity as in (5). If $j = j(x)$ tends to infinity with x, while also satisfying the bound $j(x) \leq \log_2 x - (\log_2 x)^{1/2+\epsilon}$, then, for any real t as $x \to \infty$,

$$\frac{1}{x}\#\left\{n \leq x : \frac{\log_2 p_j(n) - j}{\sqrt{j}} < t\right\} = \Phi(t) + o(1).$$

We can compare this to the result (1) from [8] (as well as our strengthening in Corollary 2.4). Since most integers n have approximately $\log_2 n$ prime factors, the α-positioned prime factor typically has index about $j = \alpha\log\log n$. Thus, comparing the α-positioned prime factor $P^{(\alpha)}(n)$, to the $j = \lceil\alpha\log_2 n\rceil$-th prime factor $p_j(n)$, we note that both $\log_2 P^{(\alpha)}(n)$ and $p_j(n)$ are distributed according to the Gaussian law around $\alpha\log_2 x$, the former has a smaller standard deviation by a factor of $\sqrt{1-\alpha}$.

[3]Here we consider $p_j(n) = \infty$ if $j > \omega(n)$.

Galambos and De Koninck [6] and later Granville [15] investigate further the sequence of consecutive intermediate prime factors of an integer and show, for almost all integers, that the sequence $\log_2 p_j$ of intermediate prime divisors can be interpreted as a Poisson process.

Finally, we note that much more is known about the distribution of the j-th largest prime divisor $P_j(n)$ than what was mentioned above. In particular, Billingsley [4] and Knuth and Trabb Pardo [19] show that $\log P_j(n)/\log n$ is distributed as the j-th coordinate of the Poisson–Dirichlet distribution. As discussed further in our concluding remarks, our results don't encompass the full range of prime divisors–in particular they don't apply to the jth largest prime factors (our results require β bounded away from 1). It would be interesting to better understanding the transition in behavior between the intermediate prime factors and these large prime factors.

Notation and conventions: Most of our notation is standard. We continue to use $P^+(n)$ for the largest prime factor of n (with $P^+(1) = 1$) and we use $P^-(n)$ for the smallest prime factor of n (taking $P^-(1) = \infty$). We say n is y-smooth if $P^+(n) \le y$, and we call n y-rough when $P^-(n) > y$. Given a set of primes E, we use $\Omega_E(n)$ to denote the number of primes of E dividing n, counted with multiplicity; explicitly, $\Omega_E(n) := \sum_{p^k \| n,\ p\in E} k$. We also denote by $E(x)$ the sum of reciprocals of the elements of E up to x, that is, $E(x) := \sum_{p\le x,\ p\in E} 1/p$.

Implied constants in $\ll$ and O-notation may always depend on any parameter declared as "fixed". In particular, they depend on α and ϵ unless stated otherwise. For rational α, the dependence on α will come from the distance of α from 0 and 1, and the size of the denominator of α. For fixed quantities δ and ϵ, we shall write $\delta \ll_\epsilon 1$ to mean that $\delta \in (0,1)$ may be fixed to be sufficiently small in terms of ϵ (we shall be using variants of this notation with δ and ϵ replaced by other fixed parameters). We write $\log_k x$ for the k-fold iterate of the natural logarithm.

3. The exact middle prime factor

Let $\overline{M}_p(x)$ denote the number of integers $n \le x$ with $\Omega(n) \equiv 1 \pmod 2$, and whose exact middle prime factor is p. In order to present our argument in as simple a manner as possible we will first prove the following theorem, which may also be of some interest in its own right.

Theorem 3.1. *Let $\epsilon > 0$ and suppose $p \to \infty$, $\beta = \frac{\log_2 p}{\log_2 x}$. Then if either $\beta < \frac{1}{5} - \epsilon$ or $\frac{1}{5} + \epsilon < \beta < 1 - \epsilon$ we have*

$$\overline{M}_p(x) = \begin{cases} \left(1 + O_\epsilon\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right)\overline{C}_\beta \dfrac{x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}}\sqrt{\log_2 x}} & \text{if } \frac{1}{5}+\epsilon < \beta < 1-\epsilon, \\ \left(1 + O_\epsilon\left(\sqrt{\frac{\log_3 x}{\log_2 x}} + \frac{(\log_2 p)^{-1/2}}{(\log p)^{\epsilon^2}}\right)\right)\overline{C} \dfrac{x}{p(\log x)^{\frac{1}{2}-\frac{3}{2}\beta}} & \text{if } 0 < \beta < \frac{1}{5}-\epsilon, \end{cases} \tag{8}$$

where

$$\overline{C}_\beta := \frac{\exp\left(\frac{\gamma(1-2\beta)}{\sqrt{\beta(1-\beta)}}\right)}{\Gamma\left(1+\sqrt{\frac{\beta}{1-\beta}}\right)} \frac{\beta^{1/4}}{2\sqrt{\pi}(1-\beta)^{3/4}} \prod_{q \text{ prime}} \left(1-\frac{1}{q}\right)^{\sqrt{\frac{1-\beta}{\beta}}} \left(1-\frac{\sqrt{\frac{1-\beta}{\beta}}}{q}\right)^{-1},$$

$$\overline{C} := \frac{e^{\frac{3\gamma}{2}}}{4\sqrt{\pi}} \prod_{q>2 \text{ prime}} \left(1+\frac{1}{q(q-2)}\right) = 0.507851\ldots.$$

Analogously to $\overline{M}_p(x)$, we can define $\underline{M}_p(x)$ to be the count of those integers having an even number of prime factors, and whose middle prime factor is p. $\underline{M}_p(x)$ satisfies an expression identical to (2) above but with different constants $\underline{C}_\beta$ and $\underline{C}$ in place of $\overline{C}_\beta$ and $\overline{C}$ respectively, namely, $\underline{C}_\beta := \overline{C}_\beta\sqrt{\frac{1-\beta}{\beta}}$ and $\underline{C} := 2\overline{C} = 1.015703\ldots$. Summing the expressions for $\overline{M}_p(x)$ and $\underline{M}_p(x)$ gives the expression for $M_p(x)$ in Theorem 2.1.

4. Technical Preparation

Before proving our main results, we state several results which will be used in the proofs. We begin with the following consequences of the classical results of Sathe–Selberg and Nicolas concerning the distribution of numbers with a given number of prime factors (see, e.g., Theorems 6.5 and 6.6 in p. 304–305, and Exercise 217 of [25]).

Lemma 4.1. *Fix $\delta \in (0,1)$. For all sufficiently large values of x,*

$$\sum_{\substack{n \le x \\ \Omega(n)=k}} 1 \ll \frac{x}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!}$$

uniformly for positive integers $k \le (2-\delta)\log_2 x$, and

$$\sum_{\substack{n \le x \\ \Omega(n)=k}} 1 \ll \frac{x \log x}{2^k}$$

uniformly for $k \ge (2+\delta)\log_2 x$.

The following lemma belongs to the study of the 'anatomy of integers', and makes precise the claim that $\Omega_E(n) = \sum_{p^k \| n,\ p\in E} k$ is typically of size $\sum_{p\le x,\ p\in E} 1/p$, uniformly across all sets of primes E. Although the statement below is slightly more general than Lemma 3.1 in [21], the same proof goes through; also compare with Theorem 08 on p. 5 of [16].

Lemma 4.2. *Fix $\epsilon_0 \in (0,1), C_0 > 0$. Let $x \ge 3$ and let E be a nonempty set of primes with smallest element p_E. With $E(x) = \sum_{p\le x,\ p\in E} 1/p$, we have, for $1 \le y \le \min\{C_0, (1-\epsilon_0)p_E\}$,*

$$\sum_{\substack{n \le x \\ \Omega_E(n) \ge yE(x)}} 1 \ll x \exp(-E(x)\cdot Q(y)),$$

where $Q(y) := y\log y - y + 1$ and the implied constant is absolute. When $0 < y \le 1$, the same inequality holds with the $\Omega_E(n)$ condition replaced by $\Omega_E(n) \le yE(x)$.

We denote by $\Phi_k(x,y)$ the number of integers $n \le x$ whose least prime factor $P^-(n) \ge y$ and where $\Omega(n) = k$. The following two results of Alladi provide estimates for $\Phi_k(x,y)$ in different ranges of y.

Theorem 4.3 (Alladi [1], Theorem 7, see also [2])**.** *Fix $r > 0$ and set $u := \frac{\log x}{\log y}$, $\xi := \frac{k}{\log u - \gamma}$. Then uniformly for $\exp((\log_2 x)^3) \le y \le \sqrt{x}$ and $1 \le k \le r \log u$, we have*[4]

$$\Phi_k(x,y) = \frac{xe^{-\gamma\xi}}{\log x\ \Gamma(1+\xi)} \cdot \frac{(\log u)^{k-1}}{(k-1)!}\left(1 + O_r\left(\frac{1}{\sqrt{\log u}}\right)\right).$$

Theorem 4.4 (Alladi [1], Theorem 6)**.** *Fix $\epsilon \in (0,2)$ and set $\mu := \frac{k-1}{\log_2 x}$. Then uniformly for $3 \le y \le \exp\left((\log x)^{2/5}\right)$ and $1 \le k \le (2-\epsilon)\log_2 x$, we have*

$$\Phi_k(x,y) = \frac{xg(y,\mu)}{\log x\ \Gamma(1+\mu)} \cdot \frac{(\log_2 x)^{k-1}}{(k-1)!}\left(1 + O\left(\frac{k(\log_2 y)^2}{(\log_2 x)^2}\right)\right),$$

where

$$g(y,\mu) := \prod_{p<y}\left(1-\frac{1}{p}\right)^{\mu}\prod_{p\ge y}\left(1-\frac{1}{p}\right)^{\mu}\left(1-\frac{\mu}{p}\right)^{-1}.$$

The truncated sums of the exponential series (and twists thereof) will play a starring role in our arguments. The following technical result provides the groundwork for estimating such sums.

Theorem 4.5 (Norton [22], Lemmas 4.5 and 4.7)**.** *Let v, θ be positive real numbers. Then*

$$\sum_{k\le(1-\theta)v}\frac{v^k}{k!} < \frac{1}{\theta\sqrt{v(1-\theta)}}\exp\left((R(-\theta)+1)v\right) \tag{9}$$

and

$$\sum_{k\ge(1+\theta)v}\frac{v^k}{k!} < \frac{\sqrt{1+\theta}}{\theta\sqrt{2\pi v}}\exp\left((R(\theta)+1)v\right) \tag{10}$$

where $R(\theta) = \theta - (\theta+1)\log(\theta+1)$.

We will be making more frequent use of the following consequence of Norton's estimates.

Lemma 4.6. *Suppose $W > 0$ is sufficiently large and that $E = o(W^{2/3})$ as $W \to \infty$. Then*

$$\sum_{0\le k\le W-E}\frac{W^k}{k!} \ll \frac{W^{1/2}}{E}\exp\left(W - \frac{E^2}{2W}\right) \quad \text{and} \quad \sum_{k\ge W+E}\frac{W^k}{k!} \ll \frac{W^{1/2}}{E}\exp\left(W - \frac{E^2}{2W}\right).$$

We note that these estimates can be interpreted as bounds on the tail probabilities of a Poisson distribution with parameter W.

Proof. For any parameter $\theta = o(W^{-1/3})$, we have

$$R(-\theta)+1 = (1-\theta)(1-\log(1-\theta)) = (1-\theta)\left(1+\theta+\frac{\theta^2}{2}+O(\theta^3)\right) = 1 - \frac{\theta^2}{2} + O(\theta^3).$$

[4]Note that while Theorem 7 of [1] is stated only for $k \le (2-\epsilon)\log u$, the concluding remarks of that paper point out how that restriction can be weakened to the version given here.

Consequently taking $\theta = E/W = o(W^{-1/3})$, we deduce from (9) and the above estimate that

$$\sum_{0\le k\le W-E} \frac{W^k}{k!} = \sum_{k\le(1-\theta)W} \frac{W^k}{k!} \ll \frac{e^W}{\theta\sqrt{W}} \exp\left(-\frac{1}{2}\theta^2 W\right) \ll \frac{W^{1/2}}{E} \exp\left(W - \frac{E^2}{2W}\right),$$

establishing the first of the claimed estimates. The proof of the second is analogous. $\square$

We conclude this section with an estimate on certain 'twisted' versions of the truncated sums of the exponential series, which arise in the arguments of Theorem 2.2.

Lemma 4.7. *Fix $\epsilon, \alpha \in (0,1)$ with α rational. Then we have, uniformly in $\beta \in [\beta_\alpha + \epsilon, 1-\epsilon]$ and in $V, E \to \infty$ satisfying $V^{1/2} \ll E$ and $E = o(V^{2/3})$,*

$$\sum_{V-E\le k\le V+E} \frac{V^k}{k!} \chi^{\{(1-\alpha)k\}} = \rho_{\chi,\alpha} e^V \left\{1 + O\left(e^{-c_\alpha V} + \frac{V^{1/2}}{E} \exp\left(-\frac{E^2}{2V}\right)\right)\right\} \tag{11}$$

where $c_\alpha > 0$ is a constant depending only on α. With the same restrictions on α, V and E,

$$\sum_{\frac{V-E}{1-\alpha}\le k\le \frac{V+E}{1-\alpha}} \frac{V^{\lfloor(1-\alpha)k\rfloor}}{\lfloor(1-\alpha)k\rfloor!} \nu^{\{(1-\alpha)k\}} = \frac{\rho_{\nu,\alpha} e^V}{1-\alpha} \left\{1 + O\left(\sqrt{\frac{E}{V}} + \frac{V^{1/2}}{E} \exp\left(-\frac{E^2}{2V}\right)\right)\right\}. \tag{12}$$

The implied constants in the two formulae above depend at most on the distance of α from 0 and 1, and the size of the denominator of α.

The above asymptotic formulae also hold true (without any effective error term) for a fixed irrational $\alpha \in (0,1)$, and with a multiplicative error of

$$1 + O\left(\frac{E^{3/2}}{V} + \frac{\log V(\log_2 V)^{1+\epsilon}}{E^{1/2}} + \frac{V^{1/2}}{E} \exp\left(-\frac{E^2}{2V}\right)\right) \text{ for almost all } \alpha \in (0,1). \tag{13}$$

Finally, given ϵ, V, E as above and a parameter $\mathcal{E} = o(1)$, we have

$$\sum_{V-E\le k\le V+E} \frac{V^k}{k!} \chi^{\{(1-\alpha)k\}} = e^V \left\{1 + O\left(\mathcal{E} + \frac{V^{1/2}}{E} \exp\left(-\frac{E^2}{2V}\right)\right)\right\}, \tag{14}$$

uniformly in $\beta \in (\epsilon, 1-\epsilon)$ and in $\alpha \in (\beta - \mathcal{E}, \beta + \mathcal{E})$.

Proof. We first consider the case when $\alpha \in (0,1)$ is rational. We start by writing $1-\alpha = a/b$ for some coprime positive integers a, b, so that $b > 1$. Then the sum on the left hand side of (11) is equal to

$$\sum_{V-E\le k\le V+E} \frac{V^k}{k!} \chi^{\{ak/b\}} = \sum_{r \bmod b} \chi^{\{ar/b\}} \sum_{\substack{V-E\le k\le V+E\\ k\equiv r \ (\mathrm{mod}\ b)}} \frac{V^k}{k!}. \tag{15}$$

By the orthogonality of additive characters, the inner sum on k is (writing $e(x) := e^{2\pi i x}$)

$$\begin{aligned}&\frac{1}{b} \sum_{\ell \bmod b} e\left(-\frac{r\ell}{b}\right) \sum_{V-E\le k\le V+E} \frac{(Ve^{2\pi i\ell/b})^k}{k!} \\ &\qquad = \frac{1}{b} \sum_{\ell \bmod b} e\left(-\frac{r\ell}{b}\right) \exp(Ve^{2\pi i\ell/b}) + O\left(e^V \frac{V^{1/2}}{E} \exp\left(-\frac{E^2}{2V}\right)\right).\end{aligned}$$

Plugging this back into (15) and interchanging sums yields

$$\sum_{V-E\le k\le V+E}\frac{V^k}{k!}\chi^{\{(1-\alpha)k\}}$$
$$=\frac{e^V}{b}\sum_{r \bmod b}\chi^{\{ar/b\}}+\frac{1}{b}\sum_{\ell=1}^{b-1}\exp(Ve^{2\pi i\ell/b})\sum_{r \bmod b}\chi^{\{ar/b\}}e\left(-\frac{r\ell}{b}\right)+O\left(e^V\frac{V^{1/2}}{E}\exp\left(-\frac{E^2}{2V}\right)\right)$$
$$=\rho_{\chi,\alpha}e^V+\frac{1}{b}\sum_{\ell=1}^{b-1}\exp(Ve^{2\pi i\ell/b})\sum_{j=0}^{b-1}\chi^{j/b}e\left(-\frac{\overline{a}j\ell}{b}\right)+O\left(e^V\frac{V^{1/2}}{E}\exp\left(-\frac{E^2}{2V}\right)\right)$$

where $\overline{a}\in\mathbb{Z}$ denotes a multiplicative inverse of a mod b, and we have noted that as r runs over the different residues mod b, so does ar. The inner sum in the last display is $O(1)$, so that the sum over ℓ above is $\ll e^{(1-c_\alpha)V}$ with $c_\alpha := 1-\cos(2\pi/b)>0$. This completes the proof of (11).

In order to show (12), we start by setting $m=\lfloor(1-\alpha)k\rfloor$, so that $m\in(V-E-1,V+E]$ and $m/(1-\alpha)\le k<(m+1)/(1-\alpha)$. This last condition automatically implies that $k\in[(V-E)/(1-\alpha),(V+E)/(1-\alpha)]$ for all $m\in[V-E,V+E-1]$. The remaining m are of the form $V+\theta E+O(1)$ for some $\theta\in\{\pm1\}$, so that for such m,

$$h(m):=m\log V-m\log m+m-\frac{1}{2}\log m=V-\frac{1}{2}\log V-\frac{E^2}{2V}+O(1),$$

and by Stirling's formula,

$$\frac{V^m}{m!}=\exp\left(h(m)-\frac{1}{2}\log(2\pi)\right)\left(1+O\left(\frac{1}{m}\right)\right)\ll\frac{e^V}{V^{1/2}}\exp\left(-\frac{E^2}{2V}\right), \tag{16}$$

which is negligible compared to the error term in (12). Hence, up to a negligible error, the sum in (12) is equal to

$$\sum_{V-E\le m\le V+E}\frac{V^m}{m!}\sum_{\frac{m}{1-\alpha}\le k<\frac{m+1}{1-\alpha}}\nu^{\{(1-\alpha)k\}}. \tag{17}$$

Now, for any positive $A<B$, we see that

$$\sum_{A\le k<B}\nu^{\{(1-\alpha)k\}}=\sum_{r \bmod b}\nu^{\{ar/b\}}\sum_{\substack{A\le k<B\\ k\equiv r \ (\mathrm{mod}\ b)}}1=(B-A)\rho_{\nu,\alpha}+O(1). \tag{18}$$

Defining $L:=\sqrt{V/E}$, we partition the interval $[V-E,V+E]$ into $\lfloor 2E/L\rfloor$ equal length subintervals I, so that each subinterval has length $2E(2E/L+O(1))^{-1}=L+O(V/E^2)=L+O(1)$. For any subinterval I of length $L+O(1)$ and any two integers $m_1,m_2\in I$, Lagrange's Mean Value Theorem implies that there exists $m'\in[m_1,m_2]$ satisfying

$$h(m_1)-h(m_2)=(m_1-m_2)\frac{\partial h}{\partial m}\Big|_{m=m'}\ll L\left\{-\log\left(1-\frac{E}{V}\right)+\frac{1}{V}\right\}\ll\frac{EL}{V},$$

whereby another application of Stirling's formula reveals that

$$\frac{V^{m_1}}{m_1!}=\frac{V^{m_2}}{m_2!}\left(1+O\left(\frac{EL}{V}\right)\right)=\frac{V^{m_2}}{m_2!}\left(1+O\left(\sqrt{\frac{E}{V}}\right)\right). \tag{19}$$

As such, fixing some integer $m_I \in I$ and letting L_I and R_I denote the least and largest integers in I respectively, we see that the contribution of all $m \in I$ to the sum (17) is

$$\begin{aligned}\sum_{m\in I}\frac{V^m}{m!}\sum_{\frac{m}{1-\alpha}\le k<\frac{m+1}{1-\alpha}}\nu^{\{(1-\alpha)k\}} &= \frac{V^{m_I}}{m_I!}\left(1+O\left(\sqrt{\frac{E}{V}}\right)\right)\sum_{\frac{L_I}{1-\alpha}\le k<\frac{R_I+1}{1-\alpha}}\nu^{\{(1-\alpha)k\}}\\ &\overset{(18)}{=} \frac{\rho_{\nu,\alpha}}{1-\alpha}\left(1+O\left(\sqrt{\frac{E}{V}}\right)\right)L\frac{V^{m_I}}{m_I!}\\ &\overset{(19)}{=} \frac{\rho_{\nu,\alpha}}{1-\alpha}\left(1+O\left(\sqrt{\frac{E}{V}}\right)\right)\sum_{m\in I}\frac{V^m}{m!}.\end{aligned}$$

Summing this over all I, and using Lemma 4.6 to extend the sum on m, we obtain (12).

Now, given a parameter $\mathcal{E} = o(1)$, we see that $\chi = (1-\alpha)\beta/(1-\beta)\alpha = 1+O(\mathcal{E})$ uniformly for $\beta \in (\epsilon, 1-\epsilon)$ and $\alpha \in (\beta-\mathcal{E}, \beta+\mathcal{E})$, so that $\chi^{\{(1-\alpha)k\}} = \exp(\{(1-\alpha)k\}\log\chi) = 1+O(\mathcal{E})$, and Lemma 4.6 completes the proof of (14).

Finally, in order to establish the assertions corresponding to both (11) and (12) for irrational $\alpha \in (\epsilon, 1-\epsilon)$, we carry out the above argument with $L := E^{1/2}$. By (19), the sums in (11) and (17) are respectively equal to

$$\sum_I \frac{V^{k_I}}{k_I!}\left(1+O\left(\frac{E^{3/2}}{V}\right)\right)\sum_{k\in I}\chi^{\{(1-\alpha)k\}} \quad\text{and}\quad \sum_I \frac{V^{m_I}}{m_I!}\left(1+O\left(\frac{E^{3/2}}{V}\right)\right)\sum_{\frac{L_I}{1-\alpha}\le k<\frac{R_I+1}{1-\alpha}}\nu^{\{(1-\alpha)k\}},$$

where (as before) the outer sums are over the $\lfloor 2E/L\rfloor$ equal length subintervals I partitioning $[V-E, V+E]$, and k_I and m_I are some integers chosen from I.

As such, it only remains to estimate the sums $\sum_{k\in J} c^{\{(1-\alpha)k\}}$ uniformly over intervals $J \subset [V-E, V+E]$ with length $|J| \to \infty$, where $c \in \{\chi, \nu\}$. Since the sequence $\{(1-\alpha)k\}_{k=1}^\infty$ is uniformly distributed mod 1, this sum is $\sim \rho_{c,\alpha}|J|$. In fact, by Koksma's inequality (Theorem 5.4 in [17]), we see that

$$\left|\frac{1}{|J|+O(1)}\sum_{k\in J}c^{\{(1-\alpha)k\}} - \int_0^1 c^t\,\mathrm{d}t\right| \le \mathrm{Var}(t\mapsto c^t)\cdot \mathrm{Disc}(\{(1-\alpha)k : k\in J\}),$$

where $\mathrm{Var}(t\mapsto c^t)$ denotes the total variation of the function $t\mapsto c^t$ on $[0,1)$ and $\mathrm{Disc}(\{(1-\alpha)k : k\in J\})$ denotes the discrepancy of the sequence $\{(1-\alpha)k : k\in J\}$. By Khintchine's bound (Theorem 5.15 in [17]), the above discrepancy is $\ll \log L(\log_2 L)^{1+\epsilon}/L$ for almost all $\alpha \in (0,1)$, while $\mathrm{Var}(t\mapsto c^t) = |c-1| \ll 1$. Carrying out the above simplifications in reverse completes the proof of the lemma. ∎

5. Mertens' Theorem dissected

We will need the following result of Lichtman estimating the sum of reciprocals of smooth numbers with a given number of prime factors.

Theorem 5.1 (Lichtman [20, Theorem 4.1]). *Fix $\epsilon > 0$, and set $r := \frac{k}{\log_2 y}$ and*

$$\eta(z) := e^{\gamma z}\prod_p\left(1-\frac{1}{p}\right)^z\left(1-\frac{z}{p}\right)^{-1}.$$

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As $y \to \infty$, we have, uniformly for $k \le (2-\epsilon)\log_2 y$,

$$\sum_{\substack{P^+(A)\le y\\ \Omega(A)=k}} \frac{1}{A} = \eta(r)\frac{(\log_2 y)^k}{k!}\left(1+O_\epsilon\left(\frac{k}{(\log_2 y)^2}\right)\right).$$

It will also be helpful to have some upper bound for the above sums that is valid for all values of k. Such a result can be obtained by an application of Rankin's method.

Lemma 5.2. *We have uniformly for $y \ge 3$ and integers $J \ge 1$,*

$$\sum_{\substack{P^+(A)\le y\\ \Omega(A)=J}} \frac{1}{A} \ll \frac{J}{2^J}\log^2 y. \tag{20}$$

Proof. For any $0<z<2$ we have

$$\sum_{\substack{P^+(A)\le y\\ \Omega(A)=J}} \frac{1}{A} < z^{-J}\sum_{P^+(A)\le y}\frac{z^{\Omega(A)}}{A} = z^{-J}\left(1-\frac{z}{2}\right)^{-1}\prod_{3\le p\le y}\left(1-\frac{z}{p}\right)^{-1}$$

$$\ll \frac{z^{-J}}{2-z}\exp\left(z\sum_{3\le p\le y}\frac{1}{p}\right) \ll \frac{z^{-J}}{2-z}(\log y)^z.$$

Taking $z = 2-1/J$ and noting that $(1-1/2J)^{-J} \asymp 1$, we obtain the desired estimate. □

We will also need a version of Lichtman's theorem (Theorem 5.1) for "large" values of k, namely those where $k > (2+\epsilon)\log_2 y$. In fact, we show that (20) above essentially gives the correct order of magnitude (up to the factor of k in the numerator) for such k. This result can be viewed as an extension of Lichtman's result on dissecting Mertens' theorem for very large values of k. We define

$$\eta_o(z) := e^{\gamma z}2^{-z}\prod_{p>2}\left(1-\frac{1}{p}\right)^z\left(1-\frac{z}{p}\right)^{-1}$$

so that $\eta_0(z) = (1-z/2)\eta(z)$ for all $z \ne 2$.

Theorem 5.3. *Fix $\epsilon \in (0,1/2)$ and $A>1$. We have uniformly for $y \ge y_0$ and $(2+\sqrt{5\epsilon})\log_2 y \le k \le (\log y)^{1/2-\epsilon}$,*

$$\sum_{\substack{P^+(A)\le y\\ \Omega(A)=k}} \frac{1}{A} = \eta_o(2)\frac{\log^2 y}{2^k}\left(1+O\left(\frac{1}{(\log y)^\epsilon\sqrt{\log_2 y}}\right)\right). \tag{21}$$

Note that

$$\eta_o(2) = \frac{e^{2\gamma}}{4}\prod_{p>2}\left(1+\frac{1}{p(p-2)}\right) = 1.201303\ldots.$$

Proof. In what follows, let $\epsilon_1 := \sqrt{5\epsilon}$. We adapt the proof of [25, Theorem II.6.6]. The sum on the left hand side of (21) is the coefficient of z^k in the function

$$\sum_{n\colon P^+(n)\le y}\frac{z^{\Omega(n)}}{n} = \prod_{p\le y}\left(1-\frac{z}{p}\right)^{-1},$$

which is holomorphic on the disk $|z| < 2 - \epsilon/2$. As such, the sum in (21) equals

$$\frac{1}{2\pi i} \oint\limits_{|z|=2-\epsilon} \prod_{p\le y} \left(1 - \frac{z}{p}\right)^{-1} \frac{\mathrm{d}z}{z^{k+1}}.$$

By the Prime Number Theorem (with the usual La Vallée Poussin error term),

$$\prod_{p\le y} \left(1 - \frac{1}{p}\right)^{z} = \frac{e^{-\gamma z}}{(\log y)^z} \left(1 + O(\exp(-K\sqrt{\log y}))\right)$$

for some absolute constant $K > 0$. Moreover, for $|z| \le 2 - \epsilon$, we have

$$\prod_{p>y} \left(1 - \frac{1}{p}\right)^{z} \left(1 - \frac{z}{p}\right)^{-1} = \exp\left(\sum_{p>y} \left\{ z \log\left(1 - \frac{1}{p}\right) - \log\left(1 - \frac{z}{p}\right)\right\}\right) = 1 + O\left(\frac{1}{y \log y}\right).$$

Consequently,

$$\sum_{\substack{P^+(A)\le y\\ \Omega(A)=k}} \frac{1}{A} = I + O\left(\exp(-K\sqrt{\log y}) \int_{|z|=2-\epsilon} \frac{(\log y)^{\Re z}}{|z|^{k+1}} |dz|\right), \tag{22}$$

where

$$I := \frac{1}{2\pi i} \oint\limits_{|z|=2-\epsilon} \frac{\eta(z)(\log y)^z}{z^{k+1}} \, \mathrm{d}z = \eta_o(2) \frac{\log^2 y}{2^k} - \frac{1}{2\pi i} \oint\limits_{|z|=2+\epsilon_1} \frac{\eta(z)(\log y)^z}{z^{k+1}} \, \mathrm{d}z \tag{23}$$

since the residue of the function $\eta(z)(\log y)^z/z^{k+1}$ at the simple pole $z = 2$ is precisely $-\eta_o(2) \log^2 y/2^k$. The error term in (22) is $\ll (\log y)^{2-\epsilon} \exp(-K\sqrt{\log y})$, which is negligible compared to the error term in (21) since $k \le (\log y)^{1/2-\epsilon}$. Moreover, the last integral in (23) is

$$\ll \oint\limits_{|z|=2+\epsilon_1} \frac{(\log y)^{\Re z}}{|z|^{k+1}} |\mathrm{d}z| \ll \frac{1}{(2+\epsilon_1)^k} \int_0^{2\pi} \exp((2+\epsilon_1) \log_2 y \cos\theta) \, \mathrm{d}\theta \ll \frac{(\log y)^{2+\epsilon_1}}{(2+\epsilon_1)^k \sqrt{\log_2 y}},$$

where we have used the fact that $\int_0^{2\pi} e^{\lambda \cos\theta} \, \mathrm{d}\theta \ll e^{\lambda}/\sqrt{\lambda}$ for all $\lambda > 1$ (see [25, p. 302]). Finally, since $k \ge (2+\epsilon_1) \log_2 y$, the last expression in the above display is

$$\begin{aligned} &= \frac{(\log y)^{2+\epsilon_1}}{2^k \sqrt{\log_2 y}} \left(\frac{2}{2+\epsilon_1}\right)^k \le \frac{(\log y)^{2+\epsilon_1}}{2^k \sqrt{\log_2 y}} \left(\frac{2}{2+\epsilon_1}\right)^{(2+\epsilon_1)\log_2 y} \\ &\qquad = \frac{\log^2 y}{2^k \sqrt{\log_2 y}} (\log y)^{-\tau(\epsilon_1)} \ll \frac{\log^2 y}{2^k} \frac{1}{(\log y)^{\epsilon} \sqrt{\log_2 y}}, \end{aligned}$$

where we have noted that for $\epsilon \in (0, 1/2)$, we have $\epsilon_1 = \sqrt{5\epsilon} \in (0, 1.6)$, so that

$$\tau(\epsilon_1) := 2\left\{\left(1 + \frac{\epsilon_1}{2}\right) \log\left(1 + \frac{\epsilon_1}{2}\right) - \frac{\epsilon_1}{2}\right\} \ge \frac{\epsilon_1^2}{5} = \epsilon.$$

Collecting estimates completes the proof of the theorem. $\square$

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Remark 5.4. Although this shall not be essential for us, it is worth noting that the range of k in Theorem 5.3 can be extended to $(2+\epsilon_y)\log_2 y \le k \le (\log y)^{1/2-\epsilon}$ for any positive parameter $\epsilon_y < 1.6$ depending on y: In fact, we can show that in this range of k,

$$\sum_{\substack{P^+(A)\le y\\ \Omega(A)=k}} \frac{1}{A} = \eta_o(2)\frac{\log^2 y}{2^k}\left\{1+O\left(\frac{1}{\epsilon_y\sqrt{\log_2 y}}\exp\left(-\frac{\epsilon_y^2}{5}\log_2 y\right)\right)\right\},$$

where the $\epsilon_y^2/5$ may be replaced by $\epsilon_y^2/4$ if $\epsilon_y \ll (\log_2 y)^{-1/3}$ and y is sufficiently large.

In order to show this, we carry out the above argument until (23) with ϵ replaced by ϵ_y. In order to bound the corresponding analogue of the last integral in (23), we note that since $\eta(z)$ has a simple pole at $z=2$, we have $\eta(z) \ll 1/|z-2| \ll 1/||z|-2| = 1/\epsilon_y$ on the circle $|z| = 2+\epsilon_y$. The above calculations now show that this integral is

$$\ll \frac{\log^2 y}{2^k}\cdot\frac{\exp(-\tau(\epsilon_y)\log_2 y)}{\epsilon_y\sqrt{\log_2 y}} \ll \frac{\log^2 y}{2^k}\frac{1}{\epsilon_y\sqrt{\log_2 y}}\exp\left(-\frac{\epsilon_y^2}{5}\log_2 y\right),$$

if $\epsilon_y \in (0,1.6)$. If $\epsilon_y \ll (\log_2 y)^{-1/3}$ (and $y \ge y_0$), then we may note that $\tau(\epsilon_y) = (2+\epsilon_y)\log(1+\epsilon_y/2) - \epsilon_y = \epsilon_y^2/4 + O(\epsilon_y^3) = \epsilon_y^2/4 + O(1/\log_2 y)$, allowing us to replace $\epsilon_y^2/5$ by $\epsilon_y^2/4$ in the last bound in the above display. $\square$

6. Proof of Theorem 3.1

We assume throughout that $\beta < 1-\epsilon$. We start by letting $K := 1.02/\log 2 \approx 1.4715$, and note that by the second assertion in Lemma 4.1, the number of $n \le x$ divisible by p which have more than $2K\log_2 x$ prime divisors is $\ll x/p(\log x)^{1.04}$, which is negligible for our purposes. Hence, it remains to show that the asymptotic formulae claimed for $\overline{M}_p(x)$ hold true for the number $N_p(x)$ of positive integers $n \le x$ having $\Omega(n) \equiv 1 \pmod 2$, $\Omega(n) \le 2K\log_2 x$ and $p_{(\Omega(n)+1)/2}(n) = p$.

Any positive integer $n > 1$ counted in $N_p(x)$ can be uniquely written in the form $n = ApB$ for some positive integers $A \le x/p$ and $B \le x/Ap$ with $P^+(A) \le p \le P^-(B)$ and with $\Omega(A) = \Omega(B) = k$, where $\Omega(n) = 2k+1 \ge 1$. As such,

$$N_p(x) = 1 + \sum_{k\le K\log_2 x}\ \sum_{\substack{A\le x/p\\ P^+(A)\le p\\ \Omega(A)=k}}\ \sum_{\substack{B\le x/Ap\\ P^-(B)\ge p\\ \Omega(B)=k}} 1 = \sum_{0\le k\le K\log_2 x}\ \sum_{\substack{A\le x/p\\ P^+(A)\le p\\ \Omega(A)=k}} \Phi_k\left(\frac{x}{Ap}, p\right). \tag{24}$$

We first consider the case $p > \exp((\log_2 x)^3)$, or equivalently $\beta > 3\log_3 x/\log_2 x$. Since $Ap^3 \le p^{k+3} \le \exp(3K\log_2 x(\log x)^{1-\epsilon}) < x^{1/2}$, we have $p \le \sqrt{x/Ap}$, and Theorem 4.3 yields

$$N_p(x) = \sum_{k\le K\log_2 x}\ \sum_{\substack{A\le x/p\\ P^+(A)\le p\\ \Omega(A)=k}} \frac{x/Ap}{\log(x/Ap)}\frac{e^{-\gamma\xi_A}(\log u_A)^{k-1}}{\Gamma(1+\xi_A)(k-1)!}\left(1+O\left(\frac{1}{\sqrt{\log u_A}}\right)\right)$$

where $u_A := \frac{\log(x/Ap)}{\log p}$ and $\xi_A := \frac{k}{\log u_A - \gamma}$. Now since $\log(Ap) \le (k+1)\log p \ll \log_2 x \log p$, we have $\log\left(\frac{x}{Ap}\right) = \log x \left(1 + O\left(\frac{\log_2 x \log p}{\log x}\right)\right)$. Consequently, recalling that $u = \frac{\log x}{\log p}$, we obtain

$$u_A = u\left(1 + O\left(\frac{\log_2 x \log p}{\log x}\right)\right) \implies (\log u_A)^{k-1} = (\log u)^{k-1}\left(1 + O\left(\frac{\log_2 x \log p}{\log x}\right)\right),$$

where the implication above uses the fact that $\log u = (1-\beta)\log_2 x > \epsilon \log_2 x$. Likewise, $\xi_A = \xi\left(1 + O\left(\frac{\log p}{\log x}\right)\right)$ where $\xi = \frac{k}{\log u - \gamma}$, so that $e^{-\gamma\xi_A} = e^{-\gamma\xi}\left(1 + O\left(\frac{\log p}{\log x}\right)\right)$, and by Lagrange's Mean Value Theorem,

$$\Gamma(1+\xi_A) = \Gamma(1+\xi) + O\left(\frac{\log p}{\log x}\right) = \Gamma(1+\xi)\left(1 + O\left(\frac{\log p}{\log x}\right)\right).$$

Collecting estimates, we now obtain

$$N_p(x) = \frac{x}{p\log x}\left(1 + O\left(\frac{1}{\sqrt{\log_2 x}}\right)\right) \sum_{k \le K\log_2 x} \frac{e^{-\gamma\xi}}{\Gamma(1+\xi)} \frac{(\log u)^{k-1}}{(k-1)!} \sum_{\substack{P^+(A)\le p \\ \Omega(A)=k}} \frac{1}{A}, \qquad (25)$$

where we have recalled that for any $k \le K\log_2 x$, we have $p^{k+1} \le x$ for all sufficiently large x, which implies that any A counted in the inner sum above is automatically $\le x/p$.

We now write $N_p(x) = N_{p,1}(x) + N_{p,2}(x)$, where $N_{p,1}(x)$ is defined by restricting the outer sum in (25) to $k \le (2-\delta)\log_2 p = (2-\delta)\beta\log_2 x$ for some $\delta \in (0,1)$, to be chosen small enough in terms of ϵ (all our statements henceforth will be valid for all $\delta \ll_\epsilon 1$).

Estimation of $N_{p,1}(x)$: In order to estimate $N_{p,1}(x)$, we invoke Theorem 5.1 to estimate the inner sum on A. This shows that $N_{p,1}(x)$ is

$$\frac{x}{p\log x}\left(1 + O\left(\frac{1}{\sqrt{\log_2 x}} + \frac{1}{\log_2 p}\right)\right) \sum_{k \le (2-\delta)\log_2 p} \eta\left(\frac{k}{\log_2 p}\right) \frac{e^{-\gamma\xi}}{\Gamma(1+\xi)} \frac{(\log u)^{k-1}}{(k-1)!} \frac{(\log_2 p)^k}{k!}$$

$$= \frac{x}{p\log x \log u}\left(1 + O\left(\frac{1}{\sqrt{\log_2 x}} + \frac{1}{\log_2 p}\right)\right) \sum_{k \le (2-\delta)\log_2 p} \eta\left(\frac{k}{\log_2 p}\right) \frac{ke^{-\gamma\xi}}{\Gamma(1+\xi)} \binom{2k}{k} \frac{w^{2k}}{(2k)!}$$

where we have defined

$$w := \sqrt{\log u \log_2 p} = \sqrt{\beta(1-\beta)}\log_2 x,$$

the geometric mean of $\log u$ and $\log_2 p$. We will find that the values of k which are most significant in the sum above are those with $k \approx w$.

By Stirling's estimate, we see that $\binom{2k}{k} = \frac{2^{2k}}{\sqrt{\pi k}}\left(1 + O\left(\frac{1}{k}\right)\right)$; hence

$$N_{p,1}(x) = \frac{x}{\pi^{1/2} p\log x \log u}\left(1 + O\left(\frac{1}{\sqrt{\log_2 x}} + \frac{1}{\log_2 p}\right)\right)$$
$$\sum_{k \le (2-\delta)\log_2 p} \eta\left(\frac{k}{\log_2 p}\right) \frac{k^{1/2}e^{-\gamma\xi}}{\Gamma(1+\xi)} \frac{(2w)^{2k}}{(2k)!}\left(1 + O\left(\frac{1}{k}\right)\right). \qquad (26)$$

Now invoking Lemma 4.6 with $W = 2w$ and $E = 6\sqrt{w \log w} =: 2w'$, we see that

$$\sum_{k \le \min\{w-w', (2-\delta)\log_2 p\}} \eta\left(\frac{k}{\log_2 p}\right) \frac{k^{1/2} e^{-\gamma\xi}}{\Gamma(1+\xi)} \frac{(2w)^{2k}}{(2k)!}$$
$$\ll w^{1/2} \sum_{m \le 2w-2w'} \frac{(2w)^m}{m!}$$
$$\ll \frac{e^{2w}}{w^8\sqrt{\log w}} \ll \frac{(\log x)^{2\sqrt{\beta(1-\beta)}}}{(\log_2 x)^4 (\log_3 x)^4} \tag{27}$$

where we have noted that $w = \sqrt{\beta(1-\beta)}\log_2 x \gg \beta^{1/2}\log_2 x \gg \sqrt{\log_2 x \log_3 x}$ since $3\log_3 x/\log_2 x < \beta < 1-\epsilon$. This shows that the total contribution from $k \le \min\{w - w', (2-\delta)\log_2 p\}$ to the right hand side of (26) is

$$\ll \frac{1}{(\log_2 x \log_3 x)^4} \frac{x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}}\log_2 x}$$

which is negligible in comparison to our error terms in either of the two cases $\beta < 1/5-\epsilon$ or $\beta \in (1/5+\epsilon, 1-\epsilon)$. (For $\beta < 1/5-\epsilon$, we make use of the easy fact that $1 - 2\sqrt{\beta(1-\beta)} > 1/2 - 3\beta/2$ for all $\beta \in (0,1)$.) In particular, for $\beta < 1/5-\epsilon$, we have $(2-\delta)\log_2 p < w - w'$, hence the above argument shows that the count $N_{p,1}(x)$ itself is absorbed in the claimed error term.

Now if $\beta \in (1/5+\epsilon, 1-\epsilon)$, then the total contribution of $k \in (w+w', (2-\delta)\log_2 p]$ to the right hand side of (26) is, by another application of Lemma 4.6,

$$\ll \frac{x}{p\log x \log u} \sum_{w+w' < k \le (2-\delta)\log_2 p} \eta\left(\frac{k}{\log_2 p}\right) \frac{k^{1/2}e^{-\gamma\xi}}{\Gamma(1+\xi)} \frac{(2w)^{2k}}{(2k)!}$$
$$\ll \frac{x\sqrt{\log_2 p}}{p\log x \log u} \sum_{m > 2w+2w'} \frac{(2w)^m}{m!} \ll \frac{1}{(\log_2 x)^9\sqrt{\log_3 x}} \frac{x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}}\sqrt{\log_2 x}}$$

which is again negligible in comparison to the error term. (Here we have noted that $w = \sqrt{\beta(1-\beta)}\log_2 x \asymp \log_2 x$.)

In the same range of β, we find that the interval $[w-w', w+w']$ gives the main contribution to the sum in (26). For k in this range we have

$$k = w + O\left(\sqrt{w\log w}\right) = w\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right),$$

and so throughout this range we find that $\xi = \frac{k}{\log u}\left(1 - \frac{\gamma}{\log u}\right)^{-1} = \sqrt{\frac{\beta}{1-\beta}} + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)$ and $\frac{k}{\log_2 p} = \sqrt{\frac{1-\beta}{\beta}} + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)$. From this a routine calculation shows that for all such k we can write

$$\eta\left(\frac{k}{\log_2 p}\right)\frac{k^{1/2}e^{-\gamma\xi}}{\Gamma(1+\xi)} = \frac{\eta\left(\sqrt{\frac{1-\beta}{\beta}}\right)\exp\left(-\gamma\sqrt{\frac{\beta}{1-\beta}}\right)\sqrt{w}}{\Gamma\left(1+\sqrt{\frac{\beta}{1-\beta}}\right)}\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right).$$

Consequently, for $\beta \in (1/5+\epsilon, 1-\epsilon)$, the contribution of $k \in [w-w', w+w']$ to $N_{p,1}(x)$ is

$$\begin{aligned}
&\frac{x}{\sqrt{\pi} p \log x \log u}\left(1+O\left(\frac{1}{\sqrt{\log_2 x}}\right)\right) \sum_{w-w' \le k \le w+w'} \eta\left(\frac{k}{\log_2 p}\right) \frac{k^{1/2} e^{-\gamma\xi}}{\Gamma(1+\xi)} \frac{(2w)^{2k}}{(2k)!} \\
&\qquad = \frac{\eta\left(\sqrt{\frac{1-\beta}{\beta}}\right) e^{-\gamma\sqrt{\frac{\beta}{1-\beta}}}}{\sqrt{\pi}\ \Gamma\left(1+\sqrt{\frac{\beta}{1-\beta}}\right)} \frac{x\sqrt{w}}{p \log x \log u}\left(1+O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right) \sum_{w-w' \le k \le w+w'} \frac{(2w)^{2k}}{(2k)!} \\
&\qquad = \frac{\overline{C}_\beta x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}}\sqrt{\log_2 x}}\left(1+O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right)
\end{aligned}$$

where we have used Lemma 4.6 to extend the sum over all values of k and noted that $\sum_{k=0}^{\infty} \frac{(2w)^{2k}}{(2k)!} = \frac{1}{2}(e^{2w}+e^{-2w}) = \frac{e^{2w}}{2} + O(e^{-2w})$. We have thus established that

$$N_{p,1}(x) = \begin{cases} \dfrac{\overline{C}_\beta x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}}\sqrt{\log_2 x}}\left(1+O\left(\sqrt{\dfrac{\log_3 x}{\log_2 x}}\right)\right) & \text{if } \frac{1}{5}+\epsilon < \beta < 1-\epsilon, \\ O\left(\dfrac{x}{p(\log x)^{1/2-3\beta/2}(\log_2 x \log_3 x)^4}\right) & \text{if } \frac{3\log_3 x}{\log_2 x} < \beta < \frac{1}{5}-\epsilon. \end{cases} \tag{28}$$

Estimation of $N_{p,2}(x)$: We recall that

$$N_{p,2}(x) = \frac{x}{p\log x}\left(1+O\left(\frac{1}{\sqrt{\log_2 x}}\right)\right) \sum_{(2-\delta)\log_2 p < k \le K \log_2 x} \frac{e^{-\gamma\xi}}{\Gamma(1+\xi)} \frac{(\log u)^{k-1}}{(k-1)!} \sum_{\substack{P^+(A) \le p \\ \Omega(A)=k}} \frac{1}{A}.$$

For $\beta \in (1/5+\epsilon, 1-\epsilon)$, we invoke Lemma 5.2 to obtain

$$N_{p,2}(x) \ll \frac{x}{p\log x} \sum_{(2-\delta)\log_2 p < k \le K\log_2 x} \frac{(\log u)^{k-1}}{(k-1)!} \frac{k}{2^k} \log^2 p \ll \frac{x \log_2 x \log^2 p}{p \log x} \sum_{k > (2-\delta)\log_2 p} \frac{v^k}{k!}$$

where $v := \frac{1}{2}\log u \asymp \log_2 x$. Defining θ so that $v(1+\theta) = (2-\delta)\log_2 p$, we see that $\theta = \frac{2(2-\delta)}{1/\beta - 1} - 1 \asymp 1$. An application of (10) now yields

$$\begin{aligned}
N_{p,2}(x) &\ll \frac{x \log_2 x \log^2 p}{p \log x} \cdot \frac{\exp\left(v(R(\theta)+1)\right)}{\sqrt{v}} \\
&\ll \frac{x\sqrt{\log_2 x}(\log x)^{2\beta} \exp\left((\frac{1-\beta}{2})(\frac{2(2-\delta)}{1/\beta-1})(1-\log\left(\frac{2(2-\delta)}{1/\beta-1}\right)) \log_2 x \right)}{p\log x} \\
&= \frac{x\sqrt{\log_2 x}}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}+F(\beta,\delta)}},
\end{aligned} \tag{29}$$

where $F(\beta,\delta) := 2\sqrt{\beta(1-\beta)} - (4-\delta)\beta + (2-\delta)\beta\log\left(\frac{2\beta(2-\delta)}{1-\beta}\right)$.

We claim that for any $\delta \ll_\epsilon 1$, $G(\delta) := \inf_{\beta \in [1/5+\epsilon, 1-\epsilon]} F(\beta,\delta) > 0$. Indeed, since F is continuous on $[1/5+\epsilon, 1-\epsilon] \times [0,1]$, so is G on $[0,1]$; hence it suffices to show that

$G(0) = \inf_{\beta\in[1/5+\epsilon,1-\epsilon]} F(\beta,0) > 0$. But this in turn is an immediate consequence of the observation that $F\left(\frac{1}{5},0\right) = 0$ and that

$$F(\beta,0) = 2\sqrt{\beta(1-\beta)} - 4\beta + 2\beta\log\left(\frac{4\beta}{1-\beta}\right)$$

is strictly increasing for $\beta > \frac{1}{5}$. This proves our claim. As such, for any fixed $\delta \ll_\epsilon 1$ and $c = c(\epsilon,\delta) \in (0, G(\delta)/2)$, we see that $F(\beta,\delta) > 2c$ for all $\beta \in (1/5+\epsilon, 1-\epsilon)$, leading to

$$N_{p,2}(x) \ll \frac{x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}+c}}$$

for all such β.

Now suppose $3\log_3 x/\log_2 x < \beta < 1/5-\epsilon$ and set $v' = 2\sqrt{v\log v}$. We proceed as in (27) to bound the contributions of $k \in ((2-\delta)\log_2 p, v-v'] \cup [v+v', K\log_2 x]$ in $N_{p,2}(x)$. Indeed, invoking (20) to bound the sum of $1/A$, we see that this contribution is

$$\begin{aligned}
&\ll \frac{x\log^2 p}{p\log x}\sum_{(2-\delta)\log_2 p\le k\le v-v'}\frac{(\log u)^{k-1}}{(k-1)!}\frac{k}{2^k} + \frac{x\log^2 p}{p\log x}\sum_{v+v'\le k\le K\log_2 x}\frac{(\log u)^{k-1}}{(k-1)!}\frac{k}{2^k}\\
&\ll \frac{x\log_2 x\log^2 p}{p\log x}\left(\sum_{k\le v-v'}\frac{v^k}{k!} + \sum_{k\ge v+v'}\frac{v^k}{k!}\right) \ll \frac{1}{\log_2 x\sqrt{\log_3 x}}\frac{x}{p(\log x)^{1/2-3\beta/2}}
\end{aligned}$$

which is negligible in comparison to the error term. Finally we use Theorem 5.3 to estimate the contribution to $N_{p,2}(x)$ from the range $(v-v', v+v')$. To do so, we choose $\epsilon_0 < \frac{1}{2}$ sufficiently small so that $(2+\sqrt{5\epsilon_0})\log_2 p < v-v'$. For sufficiently large x, the choice $\epsilon_0 = \epsilon^2$ suffices. Hence, the sought contribution is

$$\begin{aligned}
&\frac{x}{p\log x}\left(1+O\left(\frac{1}{\sqrt{\log_2 x}}\right)\right)\sum_{v-v'<k<v+v'}\frac{e^{-\gamma\xi}}{\Gamma(1+\xi)}\frac{(\log u)^{k-1}}{(k-1)!}\sum_{\substack{P^+(A)\le p\\ \Omega(A)=k}}\frac{1}{A}\\
&\qquad = \frac{\eta_o(2)e^{-\frac{\gamma}{2}}}{2\Gamma(3/2)}\frac{x\log^2 p}{p\log x}\left(1+O\left(\sqrt{\frac{\log_3 x}{\log_2 x}} + \frac{1}{(\log p)^{\epsilon_0}\sqrt{\log_2 p}}\right)\right)\sum_{v-v'<k<v+v'}\frac{v^{k-1}}{(k-1)!},
\end{aligned}$$

where we noted that for all k in the above range, $\xi = 1/2 + O(\sqrt{\log_3 x/\log_2 x})$, so that $e^{-\gamma\xi} = e^{-\gamma/2}(1+O(\sqrt{\log_3 x/\log_2 x}))$ and $\Gamma(1+\xi) = \Gamma(3/2)(1+O(\sqrt{\log_3 x/\log_2 x}))$. The constant in the last display above is exactly $\overline{C}$, while the sum on v is $e^v(1+O(1/v^2\sqrt{\log v})) = (\log x)^{\frac{1}{2}(1-\beta)}(1+O(1/(\log_2 x)^2\sqrt{\log_3 x}))$ by Lemma 4.6. Collecting estimates, we have now shown that

$$N_{p,2}(x) = \begin{cases} O\left(\dfrac{x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}+c}}\right) & \text{if } \frac{1}{5}+\epsilon < \beta < 1-\epsilon,\\[2ex] \dfrac{\overline{C}x}{p(\log x)^{(1-3\beta)/2}}\left(1+O\left(\sqrt{\dfrac{\log_3 x}{\log_2 x}} + \dfrac{(\log_2 p)^{-1/2}}{(\log p)^{\epsilon_0}}\right)\right) & \text{if } \frac{3\log_3 x}{\log_2 x} < \beta < \frac{1}{5}-\epsilon.\end{cases} \tag{30}$$

Since $N_p(x) = N_{p,1}(x) + N_{p,2}(x)$, (28) and (30) together complete the proof of Theorem 3.1 in the cases $\frac{3\log_3 x}{\log_2 x} < \beta < 1/5-\epsilon$ and $1/5+\epsilon < \beta < 1-\epsilon$.

It remains to consider the case $\beta \le 3\log_3 x/\log_2 x$, that is, $p \le \exp((\log_2 x)^3)$. In this case, an application of Theorem 4.4 to (24) yields[5]

$$N_p(x) = \frac{x}{p\log x}\left(1 + O\left(\frac{(\log_2 p)^2}{\log_2 x}\right)\right)\sum_{k\le K\log_2 x}\frac{g(p,\mu)}{\Gamma(1+\mu)}\frac{(\log_2 x)^{k-1}}{(k-1)!}\sum_{\substack{P^+(A)\le p\\ \Omega(A)=k}}\frac{1}{A} \tag{31}$$

where we have noted, as before, that $\log(x/Ap) = \log x(1+O(\log_2 x\log p/\log x))$. Proceeding as in the case $3\log_3 x/\log_2 x < \beta < 1/5-\epsilon$, we see that the contribution of $k \le \frac{1}{2}\log_2 x - 3\sqrt{\log_2 x\log_3 x}$ and $k \ge \frac{1}{2}\log_2 x + 3\sqrt{\log_2 x\log_3 x}$ to (31) is $\ll x/p(\log x)^{1/2-2\beta}(\log_2 x)^8$ which is absorbed in the error term since $(\log x)^{\beta/2} < (\log x)^{3\log_3 x/2\log_2 x} = (\log_2 x)^{3/2} \ll (\log_2 x)^4$. Finally, the contribution of the remaining k is, by Theorem 5.3, equal to

$$\begin{aligned}
&\frac{\eta_o(2)}{2}\frac{x\log^2 p}{p\log x}\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}} + \frac{1}{(\log p)^{\epsilon_0}\sqrt{\log_2 p}}\right)\right)\\
&\qquad\times\sum_{\frac{1}{2}\log_2 x - 3\sqrt{\log_2 x\log_3 x}<k<\frac{1}{2}\log_2 x+3\sqrt{\log_2 x\log_3 x}}\frac{g(p,\mu)}{\Gamma(1+\mu)}\frac{(\frac{1}{2}\log_2 x)^{k-1}}{(k-1)!}\\
&= \frac{\eta_o(2)g(p,1/2)}{2\Gamma(3/2)}\frac{x}{p(\log x)^{1/2-2\beta}}\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}} + \frac{1}{(\log p)^{\epsilon_0}\sqrt{\log_2 p}}\right)\right)
\end{aligned}$$

by a final application of Lemma 4.6 and the observation that $\mu = 1/2 + O(\sqrt{\log_3 x/\log_2 x})$. By Mertens' Theorem, we have $g(p,1/2) = e^{-\gamma/2}(\log p)^{-1/2}(1+O(1/\log p))$, completing the proof of Theorem 3.1.

6.1. **The middle prime factor when $\Omega(n)$ is even.** Recall that for those integers having an even number of prime factors, we define the middle prime factor to be the smaller of the two possible choices. As such, in order to modify the proof of Theorem 3.1 to handle $\underline{M}_p(x)$, we note that if $\Omega(n) = 2k$, and $P^{(\frac{1}{2})}(n) = p$ then we may write $n = ApB$ where $p = p_k(n)$, $\Omega(A) = k-1$, $\Omega(B) = k$ and $P^+(A) \le p \le P^-(B)$. As such, the same arguments in Theorem 3.1 go through only by changing all the conditions $\Omega(A) = k$ to $\Omega(A) = k-1$. The only notable effect of this change in our arguments is that in the case $\beta > 3\log_3 x/\log_2 x$, the expression for $N_{p,1}(x)$ has $(\log_2 p)^{k-1}/(k-1)!$ in place of $(\log_2 p)^k/k!$, and so by a change of variable $k \mapsto k-1$, we obtain the following analogue of (26):

$$\begin{aligned}
N_{p,1}(x) &= \frac{x}{\pi^{1/2}p\log x}\left(1 + O\left(\frac{1}{\sqrt{\log_2 x}} + \frac{1}{\log_2 p}\right)\right)\\
&\qquad\times\sum_{k\le(2-\delta)\log_2 p-1}\eta\left(\frac{k}{\log_2 p}\right)\frac{k^{-1/2}e^{-\gamma\xi}}{\Gamma(1+\xi)}\frac{(2w)^{2k}}{(2k)!}\left(1+O\left(\frac{1}{k}\right)\right).
\end{aligned}$$

Here the contributions of $k \le \min\{w-w', (2-\delta)\log_2 p-1\}$ and of $k \in (w+w', (2-\delta)\log_2 p-1]$ can be bounded as before; starting with the trivial bound $k^{-1/2} \ll 1$, we see that this

[5] Here it is important that our choice of K was less than 2.

contribution is[6]

$$\ll \frac{1}{(\log_2 x)^4(\log_3 x)^5}\ \frac{x}{p(\log x)^{1-2\sqrt{\beta(1-\beta)}}\sqrt{\log_2 x}}.$$

All other sums are handled exactly as in the proof of Theorem 3.1, and the final result of having $\Omega(A) = k - 1$ in place of $\Omega(A) = k$ is that our constants $\overline{C}_\beta$ and $\overline{C}$ (that had arisen for the exact middle prime factor) are multiplied by $\sqrt{(1-\beta)/\beta}$ and 2 in the cases $\beta \in (1/5+\epsilon, 1-\epsilon)$ and $\beta \in (0, 1/5-\epsilon)$ respectively.

7. The α-positioned prime factor

We now generalize Theorem 2.1, giving a proof of the more general Theorem 2.2. The proof closely follows the proof of Theorem 3.1, and so in many places we only describe the extra ideas necessary in the more general case and elaborate only on the differences with the arguments of Theorem 3.1.

Proof of Theorem 2.2. As before, we assume throughout that $\beta < 1-\epsilon$. This time at the outset, we distinguish between the cases $\beta > 3\log_3 x/\log_2 x$ and $\beta \le 3\log_3 x/\log_2 x$, considering first the former. With $K_0 := 2.04/\log 2$, Lemma 4.1 shows that the $n \le x$ with $p|n$ having $\Omega(n) \le k_\alpha := \max\{2/\alpha, 2/(1-\alpha)\}$ or $\Omega(n) > K_0 \log_2 x$ give a negligible contribution to our count; hence it suffices to establish the claimed formulae for the number $N_p^{(\alpha)}(x)$ of $n \le x$ with $\Omega(n) \in (k_\alpha, K_0 \log_2 x]$ and $P^{(\alpha)}(n) = p$. We factor each such n with $\Omega(n) = k \in (k_\alpha, K_0 \log_2 x]$, uniquely as $n = ApB$ with $P^+(A) \le p \le P^-(B)$, $\Omega(A) = \lceil \alpha k\rceil - 1$ and $\Omega(B) = k - \lceil \alpha k\rceil = \lfloor (1-\alpha)k\rfloor$, and use Theorem 4.3 to estimate the count of B's given A. This yields

$$N_p^{(\alpha)}(x) = \left(1 + O\left(\frac{1}{\sqrt{\log_2 x}}\right)\right)\frac{x}{p\log x}\sum_{k_\alpha < k \le K_0\log_2 x}\frac{e^{-\gamma\xi}}{\Gamma(1+\xi)}\frac{(\log u)^{\lfloor(1-\alpha)k\rfloor - 1}}{(\lfloor(1-\alpha)k\rfloor - 1)!}\sum_{\substack{P^+(A)\le p\\ \Omega(A) = \lceil\alpha k\rceil - 1}}\frac{1}{A} \tag{32}$$

with $\xi = \lfloor (1-\alpha)k\rfloor/(\log u - \gamma)$. As before, we write $N_p^{(\alpha)}(x) = N_{p,1}^{(\alpha)}(x) + N_{p,2}^{(\alpha)}(x)$, where $N_{p,1}^{(\alpha)}(x)$ is defined by restricting the above sum to $k_\alpha < k \le \left(\frac{2-\delta}{\alpha}\right)\log_2 p$ for some $\delta > 0$ which will be fixed to be small enough in terms of ϵ.

Estimation of $N_{p,1}^{(\alpha)}(x)$: By Theorem 5.1,

$$N_{p,1}^{(\alpha)}(x) = \left(1 + O\left(\frac{1}{\sqrt{\log_2 x}} + \frac{1}{\log_2 p}\right)\right)\frac{x}{p\log x\log u\log_2 p}$$
$$\times \sum_{k_\alpha < k \le \left(\frac{2-\delta}{\alpha}\right)\log_2 p}\eta\left(\frac{\lceil\alpha k\rceil - 1}{\log_2 p}\right)\frac{e^{-\gamma\xi}}{\Gamma(1+\xi)}\lceil\alpha k\rceil\lfloor(1-\alpha)k\rfloor\binom{k}{\lceil\alpha k\rceil}\frac{(\log u)^{\lfloor(1-\alpha)k\rfloor}(\log_2 p)^{\lceil\alpha k\rceil}}{k!}. \tag{33}$$

[6] In order to obtain the correct power saving in $\log_2 x$ for even $\Omega(n)$, it is important to truncate the sum on k to $[w - a\sqrt{w\log w}, w + a\sqrt{w\log w}]$ for some fixed $a \ge 2$. Our choice $a := 3$ when defining w' was convenient, but larger a would work just as well.

Now since $k \geq k_\alpha$, we see that $0 \leq \frac{\lceil \alpha k\rceil - \alpha k}{\alpha k} \leq \frac{1}{2}$. Consequently,

$$\lceil \alpha k\rceil \log\lceil \alpha k\rceil - \lceil \alpha k\rceil \log(\alpha k) = \alpha k\left(1+\frac{\lceil \alpha k\rceil - \alpha k}{\alpha k}\right)\log\left(1+\frac{\lceil \alpha k\rceil - \alpha k}{\alpha k}\right) = \lceil \alpha k\rceil - \alpha k + O\left(\frac{1}{k}\right)$$

and likewise

$$\lfloor (1-\alpha)k\rfloor \log\lfloor (1-\alpha)k\rfloor - \lfloor (1-\alpha)k\rfloor \log((1-\alpha)k) = \lfloor (1-\alpha)k\rfloor - (1-\alpha)k + O\left(\frac{1}{k}\right),$$

so that, by Stirling's estimate,

$$\binom{k}{\lceil \alpha k\rceil} = \frac{k^k\left(2\pi\alpha(1-\alpha)k\right)^{-1/2}\left(1+O\left(\frac{1}{k}\right)\right)}{\lceil \alpha k\rceil^{\lceil \alpha k\rceil}\lfloor (1-\alpha)k\rfloor^{\lfloor (1-\alpha)k\rfloor}} = \frac{\left(2\pi\alpha(1-\alpha)k\right)^{-1/2}}{\alpha^{\lceil \alpha k\rceil}(1-\alpha)^{\lfloor (1-\alpha)k\rfloor}}\left(1+O\left(\frac{1}{k}\right)\right).$$

Consequently, (33) yields

$$N_{p,1}^{(\alpha)}(x) = \left(1+O\left(\frac{1}{\sqrt{\log_2 x}}+\frac{1}{\log_2 p}\right)\right)\frac{1}{\sqrt{2\pi\alpha(1-\alpha)}}\frac{x}{p\log x\log u\log_2 p}$$
$$\sum_{k_\alpha<k\leq\left(\frac{2-\delta}{\alpha}\right)\log_2 p}\eta\left(\frac{\lceil \alpha k\rceil-1}{\log_2 p}\right)\frac{e^{-\gamma\xi}}{\Gamma(1+\xi)}\frac{\lceil \alpha k\rceil\lfloor (1-\alpha)k\rfloor}{k^{1/2}}\frac{w^k}{k!}\chi^{\{(1-\alpha)k\}}\left(1+O\left(\frac{1}{k}\right)\right) \quad (34)$$

where we set $w \coloneqq Q_{\alpha,\beta}\log_2 x$ with $Q_{\alpha,\beta} \coloneqq \left(\frac{\beta}{\alpha}\right)^\alpha\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha}$. Note $w \gg \beta^\alpha \log_2 x \gg \log_2 p$.

Letting $E_w \coloneqq 6\sqrt{w\log w}$, we see that the total contribution of the terms $k \leq \min\Big\{w-E_w, \left(\frac{2-\delta}{\alpha}\right)\log_2 p\Big\}$ to the expression in (34) is, by Lemma 4.6,

$$\ll \frac{x\sqrt{\log_2 p}}{p\log x\log u}\sum_{k<w-E_w}\frac{w^k}{k!} \ll \frac{x\sqrt{\log_2 p}}{p(\log x)^{1-Q_{\alpha,\beta}}\log_2 x}\frac{1}{(\log_2 p)^{18}} \quad (35)$$

which is negligible in comparison to all the claimed error terms as $1-Q_{\alpha,\beta} \geq 1-2\beta-\nu^\alpha(1-\beta)$ for all $\alpha,\beta \in (0,1)$.[7] In particular, for $\beta < \beta_\alpha - \epsilon$, this shows that $N_{p,1}^{(\alpha)}(x)$ is absorbed in the error term. On the other hand, in the case $\beta \in (\beta_\alpha+\epsilon, 1-\epsilon)$, the contribution of $k \in (w+E_w, \left(\frac{2-\delta}{\alpha}\right)\log_2 p]$ is bounded by the same expression as in (35), and thus is also negligible. Finally, in the same range of β and for any $k \in [w-E_w, w+E_w]$, we have $k = w + O(E_w)$; hence, calculations analogous to those carried out for the exact middle prime factor reveal that the contribution of all such k to the sum in (34) is

$$\left(1+O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right)\frac{C_{\beta,\alpha}}{\rho_{\chi,\alpha}}\frac{x}{p\log x\sqrt{\log_2 x}}\sum_{w-E_w\leq k\leq w+E_w}\frac{w^k}{k!}\chi^{\{(1-\alpha)k\}}. \quad (36)$$

The corresponding assertions of Lemma 4.7 now show that in the case $\beta \in (\beta_\alpha+\epsilon, 1-\epsilon)$, $N_{p,1}^{(\alpha)}(x)$ satisfies the claimed asymptotic formulae for $N_p^{(\alpha)}(x)$. The same estimate for $N_p^{(\alpha)}(x)$

[7]This follows from the observation that for each value of $\alpha \in (0,1)$, the function $H_\alpha(\beta) \coloneqq 2\beta + \nu^\alpha(1-\beta) - \left(\frac{\beta}{\alpha}\right)^\alpha\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha} \geq 0$ for all $\beta \in (0,1)$. Indeed for all such β, $\frac{\partial^2 H_\alpha}{\partial \beta^2} > 0$, so $H_\alpha(\beta)$ is convex on $(0,1)$ and has a unique minimum in $(0,1)$. Since $\frac{\partial H_\alpha}{\partial \beta}\big|_{\beta=\beta_\alpha} = 0 = H_\alpha(\beta_\alpha)$, it follows that $H_\alpha(\beta) \geq H_\alpha(\beta_\alpha) = 0$.

holds true uniformly for $\beta \in (\epsilon, 1-\epsilon)$ and $\alpha \in (\beta-\mathcal{E}, \beta+\mathcal{E})$, in which case, for all sufficiently large x, we have $\alpha > \epsilon/2$ and $\beta > \beta_\alpha + \epsilon_1/2$ for some $\epsilon_1 > 0$ depending only on ϵ. Moreover in this case, we see that $C_{\beta,\alpha}/\rho_{\chi,\alpha} = C_{\beta,\beta}(1 + O(\mathcal{E}))$, since $C_{\beta,\alpha} = \rho_{\chi,\alpha} H(\alpha, \beta)$ for some function $H(\alpha, \beta)$ differentiable on the compact set $[\epsilon/2, 1-\epsilon/2] \times [\epsilon, 1-\epsilon]$.

Estimation of $N_{p,2}^{(\alpha)}(x)$: To finish off the case $\beta > 3\log_3 x/\log_2 x$, it remains to show that $N_{p,2}^{(\alpha)}(x)$ is negligible for $\beta > \beta_\alpha + \epsilon$ and that it satisfies the claimed asymptotic formulae for $M_p^{(\alpha)}(x)$ when $\beta \in (3\log_3 x/\log_2 x, \beta_\alpha - \epsilon)$. Indeed, for $\beta > \beta_\alpha + \epsilon$, (20) shows that

$$\begin{aligned} N_{p,2}^{(\alpha)}(x) &\ll \frac{x\log^2 p}{p\log x} \sum_{\left(\frac{2-\delta}{\alpha}\right)\log_2 p < k \le K_0 \log_2 x} \frac{k}{2^{\alpha k}} \cdot \frac{(\log u)^{\lfloor (1-\alpha)k \rfloor - 1}}{(\lfloor (1-\alpha)k \rfloor - 1)!} \\ &\ll \frac{x\log_2 x}{p(\log x)^{1-2\beta}} \sum_{\left(\frac{2-\delta}{\alpha}\right)\log_2 p < k \le K_0 \log_2 x} \frac{1}{2^{\alpha k}} \cdot \frac{(\log u)^{\lfloor (1-\alpha)k \rfloor + 1}}{(\lfloor (1-\alpha)k \rfloor + 1)!} \\ &\ll \frac{x\log_2 x}{p(\log x)^{1-2\beta}} \sum_{m > \left(\frac{2-\delta}{\alpha}\right)(1-\alpha)\log_2 p} \frac{v^m}{m!} \end{aligned}$$

where we have defined $v := \nu^\alpha \log u = \nu^\alpha(1-\beta)\log_2 x$ and set $m = \lfloor (1-\alpha)k \rfloor + 1$ in the last equality above. (Here it is important that there are $\le 1/(1-\alpha) \ll 1$ many values of k giving rise to a value of m.) Considering $\theta \in (0, 1)$ satisfying $(1 + \theta)v := \left(\frac{2-\delta}{\alpha}\right)(1-\alpha)\log_2 p$, an application of (10) reveals (by a calculation analogous to (29)) that $N_{p,2}^{(\alpha)}(x) \ll x\sqrt{\log_2 x}/p(\log x)^{1-Q_{\alpha,\beta}+F(\alpha,\beta,\delta)}$, where

$$\begin{aligned} F(\alpha, \beta, \delta) := \left(\frac{\beta}{\alpha}\right)^\alpha \left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha} - 2\beta \\ - (2-\delta)\beta\left(\frac{1}{\alpha} - 1\right)\left\{1 - \log\left(\left(1 - \frac{\delta}{2}\right)\frac{2^{1/(1-\alpha)}(1/\alpha - 1)}{1/\beta - 1}\right)\right\}. \end{aligned}$$

As such, it suffices to show that for any fixed $\epsilon_1 > 0$, there exists $\epsilon_2 > 0$ (depending at most on ϵ and ϵ_1) such that $G(\delta) := \inf_{(\alpha,\beta)\in[\epsilon_1, 1-\epsilon_1]\times[\beta_\alpha+\epsilon, 1-\epsilon]} F(\alpha, \beta, \delta) > 2\epsilon_2$ for all $\delta \ll_{\epsilon,\epsilon_1} 1$.[8] But since F is continuous on $[\epsilon_1, 1-\epsilon_1] \times [\beta_\alpha+\epsilon, 1-\epsilon] \times [0, 1]$, so is G, and it suffices to show that $G(0) > 0$ or (by compactness) that $F(\alpha, \beta, 0) > 0$ for each $(\alpha, \beta) \in [\epsilon_1, 1-\epsilon_1] \times [\beta_\alpha+\epsilon, 1-\epsilon]$. This in turn follows from an analysis of the first two partial derivatives of F with respect to β on the interval $(\beta_\alpha, 1)$.[9]

Coming to the case $\beta \in (3\log_3 x/\log_2 x, \beta_\alpha - \epsilon)$, a straightforward adaptation of the prior computations (invoking Lemmas 5.2 and 4.6) shows, with $E_v := 6\sqrt{v\log v}$, the contribution of $k \in \left(\left(\frac{2-\delta}{\alpha}\right)\log_2 p, \frac{v-E_v}{1-\alpha}\right) \cup \left(\frac{v+E_v}{1-\alpha}, K_0\log_2 x\right]$ to $N_{p,2}^{(\alpha)}(x)$ is $\ll x/p(\log x)^{1-2\beta-\nu^\alpha(1-\beta)}(\log_2 x)^{17}$, which is negligible in comparison to the claimed error (here we have noted that $v \asymp \log_2 x$). On the other hand, since $\beta < \beta_\alpha - \epsilon$, we have $\frac{v-E_v}{1-\alpha} > \left(\frac{2+\epsilon_1}{\alpha}\right)\log_2 p$ for any $\epsilon_1 \ll_\epsilon 1$.

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[8] Notice that this includes both the cases of fixed and varying α considered in Theorem 2.2.

[9] Indeed, it is easy to see that for each value of $\alpha \in (0, 1)$, we have $\frac{\partial^2 F}{\partial \beta^2} > 0$ for all $\beta \in (\beta_\alpha, 1)$. Hence, $\frac{\partial F}{\partial \beta}$ an increasing function of β on $(\beta_\alpha, 1)$. Since $\frac{\partial F}{\partial \beta}\big|_{\beta=\beta_\alpha} = 0$, it follows that $F(\alpha, \beta, 0)$ itself is an increasing function of β on $(\beta_\alpha, 1)$. In particular, we have $F(\alpha, \beta, 0) > F(\alpha, \beta_\alpha, 0) = 0$, as desired.

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Consequently, (21) shows that the contribution to $N_{p,2}^{(\alpha)}(x)$ from $k \in \left[\frac{v-E_v}{1-\alpha}, \frac{v+E_v}{1-\alpha}\right]$ is

$$\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}} + \frac{(\log_2 p)^{-1/2}}{(\log p)^{\epsilon^2}}\right)\right) \frac{2\eta_o(2)\nu^\alpha \exp(-\gamma\nu^\alpha)}{\Gamma(1+\nu^\alpha)} \frac{x}{p(\log x)^{1-2\beta}} \cdot S(v), \tag{37}$$

where the sum $S(v) := \sum_{\frac{v-E_v}{1-\alpha} \le k \le \frac{v+E_v}{1-\alpha}} \frac{v^{\lfloor(1-\alpha)k\rfloor}}{\lfloor(1-\alpha)k\rfloor!} \nu^{\{(1-\alpha)k\}}$ is estimated by the corresponding assertions of Lemma 4.7. This completes the proof of the theorem for $\beta > 3\log_3 x/\log_2 x$.

It remains to consider the case $\beta \le 3\log_3 x/\log_2 x$. This time we first fix $\epsilon_1, \epsilon_2 \in (0, 1/2)$ which satisfy $(2-\epsilon_1)/(1-\alpha) > 2+\epsilon_2$,[10] and remove the $n \le x$ divisible by p which have $\Omega(n) > K_\alpha \log_2 x$ with $K_\alpha := (2-\epsilon_1)/(1-\alpha)$; the number of such n is $\ll x/p(\log x)^{K_\alpha \log 2 - 1}$, which is negligible in comparison to the error terms claimed for $\beta < \beta_\alpha - \epsilon$ (here, the constraint $\epsilon_1 < 1/2$ ensures that $K_\alpha \log 2 - 1 - (1 - 2\beta - 2\nu(1-\beta)) > \epsilon'$ for some $\epsilon' \ll_\epsilon 1$). Hence, it suffices to show the claimed asymptotics for the number $\widetilde{N}_p^{(\alpha)}(x)$ of $n \le x$ having $\Omega(n) \le K_\alpha \log_2 x$ and $P^{(\alpha)}(n) = p$. Writing each such n as ApB exactly as before, Theorem 4.4 allows us to estimate the number of B given A (in order to apply the theorem, it is crucial that $(1-\alpha)K_\alpha$ is less than and bounded away from 2). We deduce that

$$\widetilde{N}_p^{(\alpha)}(x) = \left(1 + O\left(\frac{(\log_2 p)^2}{\log_2 x}\right)\right) \frac{x}{p\log x} \sum_{k \le K_\alpha \log_2 x} \frac{g(p,\mu)}{\Gamma(1+\mu)} \frac{(\log_2 x)^{\lfloor(1-\alpha)k\rfloor - 1}}{(\lfloor(1-\alpha)k\rfloor - 1)!} \sum_{\substack{P^+(A) \le p \\ \Omega(A) = \lceil\alpha k\rceil - 1}} \frac{1}{A} \tag{38}$$

with $\mu := (\lfloor(1-\alpha)k\rfloor - 1)/\log_2 x$. Setting $v' := \nu^\alpha \log_2 x \asymp \log_2 x$ and $E' := 6\sqrt{v' \log v'}$, analogous calculations as before show that the $k < (v'-E')/(1-\alpha)$ and $k > (v'+E')/(1-\alpha)$ give a contribution $\ll x/p(\log x)^{1-2\beta-\nu^\alpha}(\log_2 x)^{17}$ to the sum in (38), which is absorbed in the error terms since $(\log x)^{\nu^\alpha \beta} < (\log x)^{3\log_3 x/\log_2 x} = (\log_2 x)^3 \ll (\log_2 x)^5$. Finally, since $\beta = o(1)$, we have $\alpha k > (2+\epsilon_1)\log_2 p$ for all $k \in \left[\frac{v'-E'}{1-\alpha}, \frac{v'+E'}{1-\alpha}\right]$. An application of Theorem 5.3 reveals that the contribution of such k to the sum (38) is equal to

$$\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}} + \frac{(\log_2 p)^{-1/2}}{(\log p)^{\epsilon^2}}\right)\right) \frac{2\eta_o(2)\nu^\alpha g(p,\nu^\alpha)}{\Gamma(1+\nu^\alpha)} \frac{x}{p(\log x)^{1-2\beta}} \cdot S(v'),$$

where the sum $S(v') := \sum_{\frac{v'-E'}{1-\alpha} \le k \le \frac{v'+E'}{1-\alpha}} \frac{v'^{\lfloor(1-\alpha)k\rfloor}}{\lfloor(1-\alpha)k\rfloor!} \nu^{\{(1-\alpha)k\}}$ is estimated by Lemma 4.7. This concludes the proof of Theorem 2.2, upon noting that $g(p,\nu^\alpha) = \frac{\exp(-\gamma\nu^\alpha)}{(\log p)^{\nu^\alpha}} \left(1 + O\left(\frac{1}{\log p}\right)\right)$. □

8. Proof of Theorem 2.3

We claim the following bounds, which together imply the assertion of the theorem.

(i) Fix $\delta \in (0, \epsilon)$. We have uniformly for $\beta \in (\epsilon, 1-\epsilon)$ and $\alpha \in (0, \epsilon-\delta) \cup (1-\epsilon+\delta, 1)$,

$$M_p^{(\alpha)}(x) \ll \frac{x}{p(\log x)^c} \tag{39}$$

[10] It is clear that such $\epsilon_1, \epsilon_2 > 0$ can be fixed only in terms of α if α itself is fixed, or only in terms of ϵ if $\alpha \in (\epsilon/2, 1-\epsilon/2)$.

for some constant $c = c(\epsilon, \delta) > 0$.[11]

(ii) Fix $\delta, \delta_0 > 0$ satisfying

$$0 < \delta \le \frac{1}{2}\left(1 - \beta_{\epsilon/8} - \frac{1}{2 - 2\nu_{\epsilon/8}}\right) < \epsilon \text{ and } 0 < \delta_0 \le 1 - 2\nu_{\epsilon/8} - (\beta_{\epsilon/8} + \delta)(2 - 2\nu_{\epsilon/8}),$$

where $\nu_{\epsilon/8} := 2^{-\frac{1}{1-\frac{\epsilon}{8}}} = 2^{-\frac{8}{8-\epsilon}}$. Then we have, uniformly for $\alpha \in \left(\frac{\epsilon}{4}, 1 - \frac{\epsilon}{4}\right)$,

$$M_p^{(\alpha)}(x) \ll \begin{cases} \dfrac{x}{p(\log x)^{1-\left(\frac{\beta}{\alpha}\right)^{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha}}\sqrt{\log_2 x}} & \text{uniformly in } \beta \in (\beta_\alpha + \delta, 1 - \delta), \\ \dfrac{x \log_2 x}{p(\log x)^{\delta_0}} & \text{uniformly in } \beta \in (\delta, \beta_\alpha + \delta]. \end{cases} \tag{40}$$

Theorem 2.3 follows by invoking (i) for $\alpha \in (0, \frac{\epsilon}{3}) \cup (1 - \frac{\epsilon}{3}, 1)$ along with (ii) for $\alpha \in (\frac{\epsilon}{4}, 1 - \frac{\epsilon}{4})$. Hence, it remains to prove (i) and (ii).

To show claim (i), we first remove all n with $|\Omega(n) - \log_2 x| \ge (\delta/4)\log_2 x$, noting that the number of such n is $\ll x/p(\log x)^{c_1}$ for some constant $c_1 = c_1(\delta) > 0$. Indeed, with E_0 denoting the set of all primes, Mertens' second theorem shows that $E_0(x/p) = \sum_{\ell \le x/p} 1/\ell = \log_2 x + C_0 + o(1)$ for some absolute constant $C_0 > 0$. As such, any n with $\Omega(n) \le (1 - \delta/4)\log_2 x$ or $\Omega(n) \ge (1 + \delta/4)\log_2 x$ can be written as $n = mp$ for some $m \le x/p$ either having $\Omega(m) \le (1 - \delta/4)E_0(x/p)$ or having $\Omega(m) \ge (1 + \delta/5)E_0(x/p)$, hence Lemma 4.2 shows that

$$\sum_{\substack{n \le x : p | n \\ \Omega(n) \le (1-\delta/4)\log_2 x}} 1 + \sum_{\substack{n \le x : p | n \\ \Omega(n) \ge (1+\delta/4)\log_2 x}} 1 \le \sum_{\substack{m \le x/p \\ \Omega(m) \le (1-\delta/4)E_0(x/p)}} 1 + \sum_{\substack{m \le x/p \\ \Omega(m) \ge (1+\delta/5)E_0(x/p)}} 1 \ll \frac{x}{p(\log x)^{c_1}}.$$

It thus remains to show that (39) holds true for the count of $n \le x$ having $P^{(\alpha)}(n) = p$ and $|\Omega(n) - \log_2 x| < (\delta/4)\log_2 x$; in the rest of the proof of claim (i), we only consider such n.

Suppose first that $\alpha \in (0, \epsilon - \delta)$. Then, since $p > \exp((\log x)^\epsilon)$, any such n has at least $\lfloor (1-\alpha)\Omega(n) \rfloor + 1 > (1-\alpha)\Omega(n) > (1-\epsilon+\delta)(1-\delta/4)\log_2 x > (1-\epsilon+\delta)(1-\delta/2)\log_2 x + 1$ many prime divisors (counted with multiplicity) greater than $\exp((\log x)^\epsilon)$. Hence, any such n can be written as $n = mp$ for some $m \le x/p$ having $\Omega_E(m) > (1-\epsilon+\delta)(1-\delta/2)\log_2 x$, where E denotes the set of primes exceeding $\exp((\log x)^\epsilon)$. Since $E(x/p) = \sum_{\exp((\log x)^\epsilon) < \ell \le x/p} 1/\ell = (1-\epsilon)\log_2 x + o(1) < (1 - \epsilon + \delta/4)\log_2 x$, we obtain, by defining $\mu_\epsilon := \frac{(1-\epsilon+\delta)(1-\delta/2)}{1-\epsilon+\delta/4} > 1$ and again applying Lemma 4.2,

$$\sum_{\substack{n \le x : P^{(\alpha)}(n) = p \\ \Omega(n) > (1-\delta/4)\log_2 x}} 1 \ \le \sum_{\substack{m \le x/p \\ \Omega_E(m) > \mu_\epsilon E(x/p)}} 1 \ \ll \ \frac{x}{p(\log x)^{c_1}}$$

for some constant $c_1 = c_1(\epsilon, \delta) > 0$. This shows claim (i) for all $\alpha \in (0, \epsilon - \delta)$.

Likewise for $\alpha \in (1 - \epsilon + \delta, 1)$, since $p < \exp((\log x)^{1-\epsilon})$, any n with $\Omega(n) < (1 + \delta/4)\log_2 x$ that is counted in $M_p^{(\alpha)}(x)$ has at most $\lfloor (1-\alpha)\Omega(n) \rfloor < (\epsilon - \delta)(1 + \delta/4)\log_2 x$ many prime divisors (counting multiplicity) greater than $\exp((\log x)^{1-\epsilon})$. Denoting by E' the set of such

[11] In the rest of this section, we shall write $C(\epsilon, \delta)$ to mean a constant C depending on ϵ and δ.

primes, we see that $E'(x/p) > (\epsilon - \delta/4)\log_2 x$. Consequently, with $\nu_\epsilon := \frac{(\epsilon-\delta)(1+\delta/4)}{\epsilon-\delta/4} \in (0,1)$, Lemma 4.2 yields

$$\sum_{\substack{n\le x: P^{(\alpha)}(n)=p\\ \Omega(n)<(1+\delta/4)\log_2 x}} 1 \le \sum_{\substack{m\le x/p\\ \Omega_{E'}(m)<\nu_\epsilon E'(x/p)}} 1 \ll \frac{x}{p(\log x)^{c_2}}$$

for some constant $c_2 = c_2(\epsilon,\delta) > 0$. This completes the proof of claim (i).

We now establish claim (ii) by closely following the proof of Theorem 2.2. To begin, defining $K_0 := 2.04/\log 2$ and $k_\epsilon := 8/\epsilon = \max\{8/\epsilon, 8/(4-\epsilon)\}$, Lemma 4.1 again shows that the contribution of n with $\Omega(n) \le k_\epsilon$ or $\Omega(n) > K_0 \log_2 x$ are both negligible, making it sufficient to show the claim with $M_p^{(\alpha)}(x)$ replaced by the count $N_p^{(\alpha)}(x)$ of $n \le x$ having $P^{(\alpha)}(n) = p$ and $\Omega(n) \in (k_\epsilon, K_0 \log_2 x]$. Since $\alpha \in (\frac{\epsilon}{4}, 1-\frac{\epsilon}{4})$, proceeding as in Theorem 2.2, we obtain

$$N_p^{(\alpha)}(x) = \left(1 + O\left(\frac{1}{\sqrt{\log_2 x}}\right)\right)\frac{x}{p\log x}\sum_{k_\epsilon<k\le K_0\log_2 x}\frac{e^{-\gamma\xi}}{\Gamma(1+\xi)}\frac{(\log u)^{\lfloor(1-\alpha)k\rfloor-1}}{(\lfloor(1-\alpha)k\rfloor-1)!}\sum_{\substack{P^+(A)\le p\\ \Omega(A)=\lceil\alpha k\rceil-1}}\frac{1}{A}.$$

Bounding the sum on A by Lemma 5.2 and proceeding as before, we see that, uniformly for $\alpha \in (\frac{\epsilon}{4}, 1-\frac{\epsilon}{4})$ and $\beta \in (\epsilon, 1-\epsilon)$, we have

$$\begin{aligned}
N_p^{(\alpha)}(x) &\ll \frac{x}{p\log x}\sum_{k_\epsilon<k\le K_0\log_2 x}\frac{k}{2^{\alpha k}}\log^2 p\cdot\frac{(\log u)^{\lfloor(1-\alpha)k\rfloor-1}}{(\lfloor(1-\alpha)k\rfloor-1)!}\\
&\ll \frac{x\log_2 x}{p(\log x)^{1-2\beta}}\sum_{k_\epsilon<k\le K_0\log_2 x}\frac{1}{2^{\alpha k}}\cdot\frac{(\log u)^{\lfloor(1-\alpha)k\rfloor}}{\lfloor(1-\alpha)k\rfloor!}\\
&\ll \frac{x\log_2 x}{p(\log x)^{1-2\beta}}\sum_{m\ge1}\frac{(2\nu_\alpha\log u)^m}{m!} \ll \frac{x\log_2 x}{p(\log x)^{1-2\beta-2\nu_\alpha(1-\beta)}},
\end{aligned}$$

where $\nu_\alpha := 2^{-1/(1-\alpha)}$ and we have set $m := \lfloor(1-\alpha)k\rfloor$, noting that there are $\le 1/(1-\alpha) \ll_\epsilon 1$ many possible values of k corresponding to a given value of m. Now with δ chosen as in the statement of the claim, we see that $\delta < \frac{\epsilon}{8} < \frac{1}{16}$, so that the function $\alpha \mapsto 1-2\nu_\alpha - (\beta_\alpha+\delta)(2-2\nu_\alpha)$ is monotonically increasing on $(\frac{\epsilon}{8}, 1-\frac{\epsilon}{8})$.[12] As such, for all $\beta \le \beta_\alpha + \delta$, the exponent of $\log x$ in the above display is $1-2\nu_\alpha-\beta(2-2\nu_\alpha) \ge 1-2\nu_\alpha-(\beta_\alpha+\delta)(2-2\nu_\alpha) > 1-2\nu_{\epsilon/8}-(\beta_{\epsilon/8}+\delta)(2-2\nu_{\epsilon/8}) \ge \delta_0$, showing the second assertion of claim (ii).

Finally, in the case $\alpha \in (\frac{\epsilon}{4}, 1-\frac{\epsilon}{4})$, $\beta \in (\beta_\alpha+\delta, 1-\delta)$, we can follow the proof of Theorem 2.2 (in the case $\beta > \beta_\alpha + \epsilon$) more closely: writing $N_p^{(\alpha)}(x) = N_{p,1}^{(\alpha)}(x) + N_{p,2}^{(\alpha)}(x)$ with the two summands defined analogously, we again see that $N_{p,2}^{(\alpha)}(x)$ is negligible in comparison to the error term, while the corresponding analogue of (34) holds true for $N_{p,1}^{(\alpha)}(x)$. However at this

[12]In fact, the function $\alpha \mapsto 1-2\nu_\alpha-(\beta_\alpha+\delta)(2-2\nu_\alpha)$ is increasing on $(0,1)$, for each fixed $\delta \in (0, 1/16)$.

point, invoking the trivial bound

$$\sum_{k_\alpha < k \le \left(\frac{2-\delta}{\alpha}\right)\log_2 p} \eta\left(\frac{\lceil \alpha k\rceil - 1}{\log_2 p}\right)\frac{e^{-\gamma\xi}}{\Gamma(1+\xi)}\frac{\lceil \alpha k\rceil \lfloor (1-\alpha)k\rfloor}{k^{1/2}}\frac{w^k}{k!}\chi^{\{(1-\alpha)k\}}$$
$$\ll (\log_2 x)^{3/2}\sum_{k\ge 1}\frac{w^k}{k!} \ll e^w(\log_2 x)^{3/2}$$

reveals that

$$N_{p,1}^{(\alpha)}(x) \ll \frac{x}{p(\log x)^{1-\left(\frac{\beta}{\alpha}\right)^\alpha\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha}}\sqrt{\log_2 x}},$$

uniformly for $\alpha \in (\frac{\epsilon}{4}, 1-\frac{\epsilon}{4})$ and $\beta \in (\beta_\alpha + \delta, 1-\delta)$. Hence the same bound holds for $N_p^{(\alpha)}(x)$ as well. This establishes our claims, and completes the proof of the theorem.

9. Proof of Theorem 2.5

We start by observing the following simple and useful bound on the tails of the Gaussian integral: for all $X > 0$, we have

$$\int_{-\infty}^{-X} e^{-u^2/2}\,\mathrm{d}u = \int_X^\infty e^{-u^2/2}\,\mathrm{d}u \le \frac{1}{X}\int_X^\infty ue^{-u^2/2}\,\mathrm{d}u = \frac{1}{X}e^{-X^2/2}. \tag{41}$$

We first prove the theorem for $-\sqrt{\log_3 x} \le t \le \sqrt{\log_3 x}$. We claim that with

$$\lambda \coloneqq \beta + \frac{t}{\sqrt{\log_2 x}},$$

the left hand side of (6) is[13]

$$\frac{p}{x}\sum_{\substack{n\le x:\, p\mid n\\ R_p(n)<\lambda}} 1 = \frac{p\log_2 x}{x}\int_0^\lambda M_p^{(\alpha)}(x)\,\mathrm{d}\alpha + O\left(\frac{1}{(\log_2 x)^{1/3}}\right). \tag{42}$$

To this end, we shall make frequent use of the following estimates

$$\sum_{\substack{n\le x:\, p\mid n\\ \Omega(n)\le \frac{1}{3}\log_2 x}} 1 \ll \frac{x}{p(\log x)^{0.15}}, \tag{43}$$

$$\sum_{\substack{n\le x:\, p\mid n\\ |\Omega(n)-\log_2 x|\ge (\log_2 x)^{2/3}}} 1 \ll \frac{x}{p(\log_2 x)^{1/3}}. \tag{44}$$

The estimate (43) is a direct consequence of Lemma 4.2 since any $n \le x$ divisible by p having $\Omega(n) \le \frac{1}{3}\log_2 x$ is of the form $n = mp$ for $m \le x/p$ having $\Omega(m) \le \frac{1}{2}\log_2(x/p)$ (for all sufficiently large x). The estimates in (44) follow from the Hardy-Ramanujan Theorem written in the form $\sum_{m\le x/p}\big(\Omega(m) - \log_2(x/p)\big)^2 \ll x\log_2 x/p$, since any n counted in those two sums is of the form mp for some $m \le x/p$ satisfying $|\Omega(m) - \log_2(x/p)| \gg (\log_2 x)^{2/3}$.

[13] In fact, the arguments in the proof show that this identity holds true uniformly for all real t, but we shall not require this.

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Turning now to the proof of the estimate (42), we first note that since the number of $n \le x$ which are divisible by p^2 is $O(x/p^2)$, it suffices to show that (42) holds true with $M_p^{(\alpha)}(x)$ replaced by the count $M_p^{(\alpha)*}(x)$ of $n \le x$ exactly divisible by p (meaning $p|n$ but $p^2 \nmid n$) for which $P^{(\alpha)}(n) = p$. Any such n has $R_p(n) = \lceil \alpha\Omega(n) \rceil/\Omega(n)$, that is, $R_p(n) - 1/\Omega(n) < \alpha \le R_p(n)$. Consequently,

$$\begin{aligned}\int_0^\lambda M_p^{(\alpha)*}(x)\,\mathrm{d}\alpha &= \sum_{\substack{n\le x:\, p\|n\\ R_p(n)<\lambda+1/\Omega(n)}} \int_{R_p(n)-1/\Omega(n)}^{\min\{\lambda, R_p(n)\}} \mathrm{d}\alpha \\ &= \sum_{\substack{n\le x:\, p\|n\\ R_p(n)<\lambda}} \frac{1}{\Omega(n)} + \sum_{\substack{n\le x:\, p\|n\\ \lambda\le R_p(n)<\lambda+1/\Omega(n)}} \left(\lambda - R_p(n) + \frac{1}{\Omega(n)}\right).\end{aligned} \tag{45}$$

For any n counted in the second sum above, we have $|R_p(n) - \lambda| < 1/\Omega(n)$, which shows that this sum is

$$\ll \sum_{\substack{n\le x:\, p\|n\\ \lambda\le R_p(n)<\lambda+1/\Omega(n)}} \frac{1}{\Omega(n)} \ll \sum_{\substack{n\le x:\, p|n\\ \Omega(n)\le\frac{1}{3}\log_2 x}} 1 + \frac{1}{\log_2 x} \sum_{\substack{n\le x:\, p\|n\\ \lambda\le R_p(n)<\lambda+1/\Omega(n)}} 1. \tag{46}$$

By (43), the first of the two sums is $\ll x/p(\log x)^{0.15}$. Any n counted in the second sum has $P^{(\lambda)}(n) = p$ and so, since $|\lambda - \beta| \le \sqrt{\frac{\log_3 x}{\log_2 x}}$, it follows from the final assertion of Theorem 2.2 (for $\alpha \in (\beta - \mathcal{E}, \beta + \mathcal{E})$), or from Theorem 2.3, that $M_p^{(\lambda)}(x) \ll x/p\sqrt{\log_2 x}$. Hence, the last expression in (46) is $\ll x/p(\log_2 x)^{3/2}$, and (45) yields

$$\int_0^\lambda M_p^{(\alpha)*}(x)\,\mathrm{d}\alpha = \sum_{\substack{n\le x:\, p\|n\\ R_p(n)<\lambda}} \frac{1}{\Omega(n)} + O\left(\frac{x}{p(\log_2 x)^{3/2}}\right) = \sum_{\substack{n\le x:\, p\|n\\ \Omega(n)\in\mathcal{W}\\ R_p(n)<\lambda}} \frac{1}{\Omega(n)} + O\left(\frac{x}{p(\log_2 x)^{4/3}}\right), \tag{47}$$

where $\mathcal{W} := (\log_2 x - (\log_2 x)^{2/3}, \log_2 x + (\log_2 x)^{2/3})$ and in the last equality above, we have invoked (43) and (44).

Finally, since any n counted in the last sum in (47) has $\Omega(n) = \log_2 x(1 + O((\log_2 x)^{-1/3}))$, it follows that the sum is

$$= \frac{1}{\log_2 x}\left(1 + O\left(\frac{1}{(\log_2 x)^{1/3}}\right)\right) \sum_{\substack{n\le x:\, p\|n\\ \Omega(n)\in\mathcal{W}\\ R_p(n)<\lambda}} 1 = \frac{1}{\log_2 x} \sum_{\substack{n\le x:\, p|n\\ R_p(n)<\lambda}} 1 + O\left(\frac{x}{p(\log_2 x)^{4/3}}\right)$$

by carrying out our earlier simplifications in reverse. Consequently, (47) yields

$$\int_0^\lambda M_p^{(\alpha)*}(x)\,\mathrm{d}\alpha = \frac{1}{\log_2 x} \sum_{\substack{n\le x:\, p|n\\ R_p(n)<\lambda}} 1 + O\left(\frac{x}{p(\log_2 x)^{4/3}}\right),$$

establishing our claim (42) uniformly for all $t \in [-\sqrt{\log_3 x}, \sqrt{\log_3 x}]$. Hence in order to complete the proof of the theorem for all t in this range, it suffices to show that

$$\int_0^{\lambda} M_p^{(\alpha)}(x)\, \mathrm{d}\alpha = \frac{x}{p \log_2 x}\left\{\Phi\left(\frac{t}{\sqrt{\beta(1-\beta)}}\right) + O\left(\frac{(\log_3 x)^{3/2}}{(\log_2 x)^{1/2}}\right)\right\} \tag{48}$$

uniformly for all such t.

Now for $t \in [-\sqrt{\log_3 x}, \sqrt{\log_3 x}]$, we have $\beta - \mathcal{E} \le \lambda \le \beta + \mathcal{E}$ where $\mathcal{E} := \sqrt{\log_3 x / \log_2 x}$. Furthermore, for all $\alpha \in (\beta - \mathcal{E}, \beta + \mathcal{E})$ we again find ourselves in the final case of Theorem 2.2, an application of which yields,

$$\int_{\beta-\mathcal{E}}^{\lambda} M_p^{(\alpha)}(x)\, \mathrm{d}\alpha = \left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right) \frac{C_{\beta,\beta} x}{p \log x \sqrt{\log_2 x}} \int_{\beta-\mathcal{E}}^{\lambda} (\log x)^{\left(\frac{\beta}{\alpha}\right)^{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha}} \mathrm{d}\alpha.$$

To analyze the integral above, we note that for all $\alpha =: \beta + \eta \in [\beta - \mathcal{E}, \beta + \mathcal{E}]$,

$$\begin{aligned}
\alpha \log\left(\frac{\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{1-\beta}{1-\alpha}\right) &= -(\beta+\eta)\log\left(1 + \frac{\eta}{\beta}\right) - (1-\beta-\eta)\log\left(1 - \frac{\eta}{1-\beta}\right) \\
&= -\frac{\eta^2}{2\beta(1-\beta)} + O(\eta^3),
\end{aligned}$$

so that $\left(\frac{\beta}{\alpha}\right)^{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha} = 1 - \frac{\eta^2}{2\beta(1-\beta)} + O(\eta^3)$, leading to

$$(\log x)^{\left(\frac{\beta}{\alpha}\right)^{\alpha}\left(\frac{1-\beta}{1-\alpha}\right)^{1-\alpha}} = (\log x)\exp\left(-\frac{\eta^2 \log_2 x}{2\beta(1-\beta)}\right)\left(1 + O\left(\frac{(\log_3 x)^{3/2}}{(\log_2 x)^{1/2}}\right)\right). \tag{49}$$

As such,

$$\begin{aligned}
\int_{\beta-\mathcal{E}}^{\lambda} M_p^{(\alpha)}(x)\, \mathrm{d}\alpha &= \left(1 + O\left(\frac{(\log_3 x)^{3/2}}{(\log_2 x)^{1/2}}\right)\right) \frac{C_{\beta,\beta} x}{p\sqrt{\log_2 x}} \int_{-\mathcal{E}}^{\lambda-\beta} \exp\left(-\frac{\eta^2 \log_2 x}{2\beta(1-\beta)}\right) \mathrm{d}\eta \\
&= \left(1 + O\left(\frac{(\log_3 x)^{3/2}}{(\log_2 x)^{1/2}}\right)\right) \frac{x}{p \log_2 x} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{\frac{\log_3 x}{\beta(1-\beta)}}}^{\frac{t}{\sqrt{\beta(1-\beta)}}} \exp\left(-\frac{\tau^2}{2}\right) \mathrm{d}\tau.
\end{aligned}$$

Invoking the bound (41), we obtain, uniformly for $t \in [-\sqrt{\log_3 x}, \sqrt{\log_3 x}]$,

$$\int_{\beta-\mathcal{E}}^{\beta + t/\sqrt{\log_2 x}} M_p^{(\alpha)}(x)\, \mathrm{d}\alpha = \frac{x}{p \log_2 x}\left\{\Phi\left(\frac{t}{\sqrt{\beta(1-\beta)}}\right) + O\left(\frac{(\log_3 x)^{3/2}}{(\log_2 x)^{1/2}}\right)\right\}, \tag{50}$$

where we have noted that $\exp\left(-\frac{\log_3 x}{2\beta(1-\beta)}\right) \ll \frac{1}{(\log_2 x)^2}$, since $\beta(1-\beta) \le 1/4$.

We now show that the contribution to the integral in (48) from α outside $[\beta - \mathcal{E}, \beta + t/\sqrt{\log_2 x}]$ is negligible. To that end, we start by noting that by an argument analogous to the above, we have

$$\int_0^1 M_p^{(\alpha)}(x)\, \mathrm{d}\alpha = \frac{x}{p \log_2 x} + O\left(\frac{x}{p(\log_2 x)^2}\right) \tag{51}$$

uniformly for $\beta \in (\epsilon, 1-\epsilon)$. Indeed, it again suffices to show the above to be true with $M_p^{(\alpha)}(x)$ replaced by $M_p^{(\alpha)*}(x)$, and by a computation similar to (45), we find that

$$\int_0^1 M_p^{(\alpha)*}(x)\,\mathrm{d}\alpha = \sum_{\substack{n\le x\\ p\| n}} \frac{1}{\Omega(n)} = \sum_{\substack{n\le x\\ p| n}} \frac{1}{\Omega(n)} + O\left(\frac{x}{p^2}\right) = \sum_{m\le x/p} \frac{1}{\Omega(m)+1} + O\left(\frac{x}{p^2}\right)$$
$$= \sum_{1<m\le x/p} \frac{1}{\Omega(m)} + O\left(\sum_{1<m\le x/p}\frac{1}{\Omega(m)^2} + \frac{x}{p^2}\right) = \frac{x}{p\log_2 x} + O\left(\frac{x}{p(\log_2 x)^2}\right)$$

where in the last equality, we have invoked Theorems 5 and 14 in [7], in the weak forms $\sum_{1<n\le x} 1/\Omega(n) = x/\log_2 x + O(x/(\log_2 x)^2)$ and $\sum_{1<n\le x} 1/\Omega(n)^2 \ll x/(\log_2 x)^2$ respectively.[14]

Comparing (50) with (51) and taking $t = \sqrt{\log_3 x}$ we obtain

$$\int_0^{\beta-\mathcal{E}} M_p^{(\alpha)}(x)\,\mathrm{d}\alpha + \int_{\beta+\mathcal{E}}^1 M_p^{(\alpha)}(x)\,\mathrm{d}\alpha$$
$$= \frac{x}{p\log_2 x}\left\{1-\Phi\left(\sqrt{\frac{\log_3 x}{\beta(1-\beta)}}\right)\right\} + O\left(\frac{x(\log_3 x)^{3/2}}{p(\log_2 x)^{3/2}}\right) \ll \frac{x(\log_3 x)^{3/2}}{p(\log_2 x)^{3/2}},$$

where in the last equality, we have again applied (41). Combining (50) with the bound above for $\int_0^{\beta-\mathcal{E}} M_p^{(\alpha)}(x)\,\mathrm{d}\alpha$ yields the desired relation (48) for all $t\in[-\sqrt{\log_3 x}, \sqrt{\log_3 x}]$, proving the theorem for t in this range.

Now suppose $t \le -\sqrt{\log_3 x}$. We claim that both sides of (6) are absorbed in the error term $O(1/(\log_2 x)^{1/3})$. This is immediate for the right hand side since by (41), we have

$$\Phi\left(\frac{t}{\sqrt{\beta(1-\beta)}}\right) \le \Phi\left(-\sqrt{\frac{\log_3 x}{\beta(1-\beta)}}\right) \ll \frac{1}{(\log_2 x)^2\sqrt{\log_3 x}}.$$

On the other hand, the left hand side of (6) is

$$\frac{p}{x}\sum_{\substack{n\le x:\, p|n\\ R_p(n)<\beta+\frac{t}{\sqrt{\log_2 x}}}} 1 \;\le\; \frac{p}{x}\sum_{\substack{n\le x:\, p|n\\ R_p(n)<\beta-\frac{\sqrt{\log_3 x}}{\sqrt{\log_2 x}}}} 1 \;=\; \Phi\left(-\sqrt{\frac{\log_3 x}{\beta(1-\beta)}}\right) + O\left(\frac{1}{(\log_2 x)^{1/3}}\right) \ll \frac{1}{(\log_2 x)^{1/3}},$$

where we have used the assertion of the theorem for $t = -\sqrt{\log_3 x}$, as established before.

Finally, for $t \ge \sqrt{\log_3 x}$, the reasoning is analogous: by (41), the right hand side of (6) is

$$1 + O\left(\frac{1}{t}\exp\left(-\frac{t^2}{2\beta(1-\beta)}\right) + \frac{1}{(\log_2 x)^{1/3}}\right)$$
$$= 1 + O\left(\frac{1}{\sqrt{\log_3 x}}\exp\left(-\frac{\log_3 x}{2\beta(1-\beta)}\right) + \frac{1}{(\log_2 x)^{1/3}}\right) = 1 + O\left(\frac{1}{(\log_2 x)^{1/3}}\right),$$

[14] Alternatively, we may replicate the arguments used to handle the sum $\sum_{n\le x:\, p\| n,\ R_p(n)<\lambda} \frac{1}{\Omega(n)}$ in (45); this gives an error term of $O(x/p(\log_2 x)^{4/3})$, which is sufficient for the theorem.

whereas by the $t = \sqrt{\log_3 x}$ case of the theorem (that we already proved),

$$1 \geq \frac{p}{x} \sum_{\substack{n \leq x:\, p|n \\ R_p(n) < \beta + \frac{t}{\sqrt{\log_2 x}}}} 1 \geq \frac{p}{x} \sum_{\substack{n \leq x:\, p|n \\ R_p(n) < \beta + \frac{\sqrt{\log_3 x}}{\sqrt{\log_2 x}}}} 1$$

$$= \Phi\left(\sqrt{\frac{\log_3 x}{\beta(1-\beta)}}\right) + O\left(\frac{1}{(\log_2 x)^{1/3}}\right) = 1 + O\left(\frac{1}{(\log_2 x)^{1/3}}\right)$$

showing that the left hand side is also $1 + O\left(\frac{1}{(\log_2 x)^{1/3}}\right)$. This completes the proof.

10. Proof of Theorem 2.6

We start by writing

$$\frac{1}{x}\sum_{n \leq x} \log P^{\left(\frac{1}{2}\right)}(n) = \frac{1}{x}\sum_{p \leq x} M_p^{\left(\frac{1}{2}\right)}(x) \log p. \tag{52}$$

The trivial bound $M_p^{\left(\frac{1}{2}\right)}(x) \leq \frac{x}{p}$ shows that the contribution to the above sum from $p \leq \exp\left(\sqrt{\log x}\right)$ is

$$\frac{1}{x} \sum_{p \leq \exp(\sqrt{\log x})} M_p^{\left(\frac{1}{2}\right)}(x) \log p < \sum_{p \leq \exp(\sqrt{\log x})} \frac{\log p}{p} \ll \sqrt{\log x},$$

and for $\sqrt{x} < p \leq x$ we have $M_p^{\left(\frac{1}{2}\right)}(x) = 1$ showing that the contribution from such p is $O(1)$. Next we bound the contribution from those p with $\exp\left((\log x)^{0.999}\right) < p \leq \sqrt{x}$.

The same arguments as in the proof of Theorem 2.3 (claim (i)) show that $M_p^{\left(\frac{1}{2}\right)}(x) \ll x/p(\log x)^{0.42}$ uniformly for $p \in (\exp((\log x)^{0.999}), \sqrt{x}]$: indeed once again, any n counted in $M_p^{\left(\frac{1}{2}\right)}(x)$ either has $\Omega(n) \leq 0.229 \log_2 x$ or has more than $\frac{0.229}{2}\log_2 x$ many prime factors (counted with multiplicity) exceeding $\exp((\log x)^{0.999})$. As such, two applications of Lemma 4.2 with the set E being the full set of primes or the set of primes exceeding $\exp((\log x)^{0.999})$ respectively, show that the contribution of both of these possibilities is $\ll x/p(\log x)^{0.42}$, as desired. Consequently, the total contribution from the primes $p \in (\exp((\log x)^{0.999}), \sqrt{x}]$ is

$$\frac{1}{x} \sum_{\exp((\log x)^{0.999}) < p \leq \sqrt{x}} M_p^{\left(\frac{1}{2}\right)}(x) \log p \ll \frac{1}{(\log x)^{0.42}} \sum_{p \leq x} \frac{\log p}{p} \ll (\log x)^{0.58},$$

which is also negligible for our purposes.

It thus remains to consider primes $\exp\left(\sqrt{\log x}\right) < p \leq \exp\left((\log x)^{0.999}\right)$, for which we can use Theorem 2.1 to estimate $M_p^{\left(\frac{1}{2}\right)}(x)$. Thus the sum (52) for p in this range is

$$\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right) \sum_{\exp\left(\sqrt{\log x}\right) < p \leq \exp((\log x)^{0.999})} C_\beta \frac{(\log x)^{\beta + 2\sqrt{\beta(1-\beta)} - 1}}{p\sqrt{\log_2 x}} \tag{53}$$

where $\beta = \frac{\log_2 p}{\log_2 x}$ is a function of p. Rewriting the sum above as an integral and using the prime number theorem we have

$$\sum_{\exp\left(\sqrt{\log x}\right)<p\leq\exp((\log x)^{0.999})} C_\beta \frac{(\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1}}{p\sqrt{\log_2 x}} = \int_{\exp\left(\sqrt{\log x}\right)}^{\exp\left((\log x)^{0.999}\right)} C_\beta \frac{(\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1}}{t\sqrt{\log_2 x}} \mathrm{d}\pi(t)$$

$$= \int_{\exp\left(\sqrt{\log x}\right)}^{\exp\left((\log x)^{0.999}\right)} C_\beta \frac{(\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1}}{t\log t\sqrt{\log_2 x}} \mathrm{d}t \;+\; O\big(\exp(-K_0(\log x)^{1/4})\big) \tag{54}$$

for some absolute constant $K_0 > 0$, where in the integrals we have $\beta := \log_2 t/\log_2 x$. Here to pass to the second line above, we have noted that writing $f(t) := C_\beta \frac{(\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1}}{t\sqrt{\log_2 x}}$ and $\mathcal{E}(t) := \pi(t) - \mathrm{li}(t) \ll t\exp(-K\sqrt{\log t})$, the function $f(t)$ is monotonically decreasing for all sufficiently large x (and for $t \geq \exp(\sqrt{\log x})$),[15] so that two applications of the Riemann-Stieltjes integration by parts yield (with $K_0 := K/2$),

$$\begin{aligned}
\int_{\exp\left(\sqrt{\log x}\right)}^{\exp\left((\log x)^{0.999}\right)} f(t)\,\mathrm{d}\mathcal{E}(t) &= -\int_{\exp\left(\sqrt{\log x}\right)}^{\exp\left((\log x)^{0.999}\right)} \mathcal{E}(t) f'(t)\,\mathrm{d}t + O\big(\exp(-K_0(\log x)^{1/4})\big) \\
&\ll \int_{\exp\left(\sqrt{\log x}\right)}^{\exp\left((\log x)^{0.999}\right)} t\exp(-K\sqrt{\log t}) f'(t)\,\mathrm{d}t + \exp(-K_0(\log x)^{1/4}) \\
&\ll \int_{\exp\left(\sqrt{\log x}\right)}^{\exp\left((\log x)^{0.999}\right)} f(t)\exp(-K\sqrt{\log t})\mathrm{d}t + \exp(-K_0(\log x)^{1/4}) \\
&\ll \exp(-K_0(\log x)^{1/4}),
\end{aligned}$$

establishing (54). Continuing from there, we find that the expression in (53) is

$$\left(1 + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right) \int_{\exp\left(\sqrt{\log x}\right)}^{\exp\left((\log x)^{0.999}\right)} C_\beta \frac{(\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1}}{t\log t\sqrt{\log_2 x}} \mathrm{d}t + O\big(\exp(-K_0(\log x)^{1/4})\big).$$

Changing the variable of integration to β, using $t = \exp\left((\log x)^\beta\right)$ we find that $\mathrm{d}t = t\log t\log_2 x\ \mathrm{d}\beta$ and so the main term above becomes

$$\left(\sqrt{\log_2 x} + O\left(\sqrt{\log_3 x}\right)\right) \int_{1/2}^{0.999} C_\beta (\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1} \mathrm{d}\beta. \tag{55}$$

We find that the exponent of $\log x$ in the integrand above achieves its maximum at $B_0 := \frac{1}{2} + \frac{1}{2\sqrt{5}}$, and its value at that point is $B_0 + 2\sqrt{B_0(1-B_0)} - 1 = \frac{\sqrt{5}-1}{2} = \varphi'$.

Furthermore, defining $\beta' := \beta - B_0$, and expanding as a Taylor series around B_0 gives

$$\beta + 2\sqrt{\beta(1-\beta)} - 1 = \varphi' - \frac{5\sqrt{5}}{4}\beta'^2 + O\left(\beta'^3\right). \tag{56}$$

[15]To see this, we write $\log f(t) =: F_1(\beta) + F_2(\beta)\log_2 x - \log t - \frac{1}{2}\log_3 x$ for certain differentiable functions F_1, F_2, and differentiate both sides of this identity with respect to t to obtain $f'(t)/f(t) < -1/t + O(1/t\log t) < 0$ for all sufficiently large x, uniformly in $\beta = \log_2 t/\log_2 x \in [0.5, 0.999]$.

Inserting this into (55) and treating first only the range $\beta \in \left[B_0 - \sqrt{\frac{\log_3 x}{\log_2 x}}, B_0 + \sqrt{\frac{\log_3 x}{\log_2 x}}\right]$, we find that

$$\begin{aligned}
&\int_{B_0-\sqrt{\frac{\log_3 x}{\log_2 x}}}^{B_0+\sqrt{\frac{\log_3 x}{\log_2 x}}} C_\beta (\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1}\,\mathrm{d}\beta \\
&\qquad = \left(C_{B_0} + O\left(\sqrt{\frac{\log_3 x}{\log_2 x}}\right)\right) \int_{-\sqrt{\frac{\log_3 x}{\log_2 x}}}^{\sqrt{\frac{\log_3 x}{\log_2 x}}} \exp\left(\left(\varphi' - \tfrac{5\sqrt{5}}{4}\beta'^2 + O\left(\beta'^3\right)\right)\log_2 x\right)\mathrm{d}\beta' \\
&\qquad = \frac{C_{B_0}(\log x)^{\varphi'}}{\sqrt{\frac{5\sqrt{5}}{4}\log_2 x}} \left(1 + O\left(\frac{(\log_3 x)^{3/2}}{\sqrt{\log_2 x}}\right)\right) \int_{-\sqrt{\frac{5\sqrt{5}}{4}\log_3 x}}^{\sqrt{\frac{5\sqrt{5}}{4}\log_3 x}} \exp\left(-\tau^2\right)\mathrm{d}\tau \\
&\qquad = C_{B_0}(\log x)^{\varphi'}\sqrt{\frac{\pi}{\frac{5\sqrt{5}}{4}\log_2 x}} \left(1 + O\left(\frac{(\log_3 x)^{3/2}}{\sqrt{\log_2 x}} + \frac{\exp\left(-\frac{5\sqrt{5}}{4}\log_3 x\right)}{\sqrt{\log_3 x}}\right)\right) \\
&\qquad = \left(C_{B_0} + O\left(\frac{(\log_3 x)^{3/2}}{\sqrt{\log_2 x}}\right)\right)(\log x)^{\varphi'}\sqrt{\frac{4\pi}{5\sqrt{5}\log_2 x}}.
\end{aligned}$$

Substituting the value of C_{B_0} from (3), we deduce that the contribution to the expression (55) from $\beta \in \left[B_0 - \sqrt{\frac{\log_3 x}{\log_2 x}},\ B_0 + \sqrt{\frac{\log_3 x}{\log_2 x}}\right]$ is

$$\left(\frac{e^{-\gamma}}{\Gamma(\varphi+1)}\frac{\varphi+1}{\sqrt{5}}\prod_p\left(1-\frac{1}{p}\right)^{\varphi'}\left(1-\frac{\varphi'}{p}\right)^{-1} + O\left(\frac{(\log_3 x)^{3/2}}{\sqrt{\log_2 x}}\right)\right)(\log x)^{\varphi'}.$$

We conclude by bounding the contribution from $\beta \in \left[\frac{1}{2}, B_0 - \sqrt{\frac{\log_3 x}{\log_2 x}}\right)$ and from $\beta \in \left(B_0 + \sqrt{\frac{\log_3 x}{\log_2 x}}, 0.999\right]$ to the expression (55). Noting that the function $\beta \mapsto \beta + 2\sqrt{\beta(1-\beta)} - 1$ is increasing on $\left[\frac{1}{2}, B_0 - \sqrt{\frac{\log_3 x}{\log_2 x}}\right)$ and then using the expansion (56), we deduce that the contribution from this interval is

$$\begin{aligned}
&\ll \sqrt{\log_2 x}\int_{1/2}^{B_0-\sqrt{\frac{\log_3 x}{\log_2 x}}} C_\beta(\log x)^{\beta+2\sqrt{\beta(1-\beta)}-1}\,\mathrm{d}\beta \\
&\qquad\qquad \ll (\log x)^{\varphi'}\exp\left(-\tfrac{5\sqrt{5}}{4}\log_3 x\right)\sqrt{\log_2 x} \ \le\ \frac{(\log x)^{\varphi'}}{(\log_2 x)^2}.
\end{aligned}$$

Finally, the contribution of the interval $\left(B_0 + \sqrt{\frac{\log_3 x}{\log_2 x}}, 0.999\right]$ can be bounded analogously, by noting that the function $\beta \mapsto \beta + 2\sqrt{\beta(1-\beta)} - 1$ is decreasing on this interval. This completes the proof of the theorem.

11. Concluding Remarks

While we were able to obtain asymptotic expressions for the frequency with which p is the middle or α-positioned prime factor of an integer up to x for a very wide range of values of p (depending on x), our theorems don't quite encompass the full range of p.

In particular, we are unable to treat those primes p for which β is too close to either β_α or to 1. (We also don't consider primes p fixed as $x \to \infty$, however this is treated, for the middle prime factor, in [12].)

The obstacles to understanding the behavior in these two remaining ranges are somewhat different. The complication when $\beta \approx \beta_\alpha$ comes from the phase transition that occurs when $k \approx 2\log_2 y$ in the asymptotics of the sums $\sum_{P^+(n)\le y,\ \Omega(n)=k} \frac{1}{n}$. (See Theorems 5.1 and 5.3.) Extending our results to β in this range would require obtaining asymptotic expressions for these sums near this phase transition. Balazard, Delange and Nicolas [3] have investigated the closely related phase transition that happens in the counting problem of $N(x,k) = \#\{n \le x \mid \Omega(n) = k\}$ for k near $2\log_2 x$, and found, for such k near this phase transition, that the correct asymptotic expression for $N(x,k)$ is a "Gaussian transition" between the asymptotic expressions valid for k bounded away on either side of this transition value. Hsien-Kuei Hwang [18] also found an alternative derivation of this transition behaviour. It seems likely that a similar transition happens in this case, however we don't investigate this any further here. Investigating large values of p when β approaches 1 appears to be more difficult. Here the obstacle is the range of validity of Alladi's result (Theorem 4.3). Extending our results would require obtaining an asymptotic expression for $\Phi_k(x,y)$ that holds for values of k that are large relative to $\log u$, where $u = \frac{\log x}{\log y}$.

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