The maximal size of the k-fold divisor function for very large k

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Abstract. Let $d_k(n)$ denote the number of ways of writing n as an (ordered) product of k positive integers. When $k = 2$, Wigert proved in 1907 that

$$(*) \qquad \log d_k(n) \le (1 + o(1)) \log k \frac{\log n}{\log\log n} \qquad (n \to \infty).$$

In 1992, Norton showed that (*) holds whenever $k = o(\log n)$; this is sharp, since (*) holds with equality when n is a product of the first several primes. In this note, we determine the maximal size of $\log d_k(n)$ when $k \gg \log n$. To illustrate: Let $\kappa > 0$ be fixed, and let $k, n \to \infty$ in such a way that $k/\log n \to \kappa$; then

$$\log d_k(n) \le \left(s + \kappa \sum_{p \text{ prime}} \sum_{\ell \ge 1} \frac{1}{\ell p^{\ell s}} + o(1)\right) \log n,$$

where $s > 1$ is implicitly defined by $\sum_{p \text{ prime}} \frac{\log p}{p^s - 1} = \frac{1}{\kappa}$. Moreover, this upper bound is optimal for every value of κ. Our results correct and improve on recent work of Fedorov.

1. Introduction.

For integers k, n with $k \ge 2$ and $n \ge 1$, we let $d_k(n)$ denote the number of ways of writing n as an ordered product of k positive integers; we abbreviate $d_2(n)$ to $d(n)$. The maximal order of $d(n)$ was first investigated by Runge [**Run85**] in 1885, who showed that $d(n) = o(n^\epsilon)$ (as $n \to \infty$) for each fixed $\epsilon > 0$. (He used this result to prove that 100% of quintic polynomials $x^5 + ux + v \in \mathbb{Z}[x]$, when ordered by height, are not solvable by radicals.) Developing Runge's method, Wigert [**Wig07**] showed in 1907 that

$$(1) \qquad \log d(n) \le (1 + o(1)) \log 2 \frac{\log n}{\log\log n}, \qquad \text{as } n \to \infty.$$

The estimate (1) is easily shown to be sharp, holding with equality when n is a product of the first several prime numbers.

Several authors have proved results analogous to (1) valid for large classes of arithmetic functions. See, for example, [**DF58, SSR75, Shi80, BP87**]. From any

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of these, one can establish for each fixed k a $d_k(n)$-analogue of (1), namely

$$\log d_k(n) \le (1+o(1)) \log k \frac{\log n}{\log\log n}, \qquad \text{as } n \to \infty.^1 \tag{2}$$

In 1992, Norton made a detailed study of the maximum size of $d_k(n)$ in various ranges of k vs. n (see [**Nor92**, Theorem 1.29]). An elegant consequence of his results is that (2) holds uniformly in the range $k = o(\log n)$. This corollary, together with the observation that (1) is sharp (shown again by considering 'primorial' values of n), is recorded as Corollary 1.36 in [**Nor92**]. Norton goes on to write (in notation changed to match ours) "We have not been able to prove a result as precise as Corollary 1.36 when $k \gg \log n$." In this note, we present sharp upper bounds on $\log d_k(n)$ in the range $k \gg \log n$ left open by Norton.

Independent investigations into the maximal size of $\log d_k(n)$, with k allowed to grow with n, have been carried out recently by Fedorov. For instance, in [**Fed13b**], Fedorov shows that (2) holds uniformly for $k = o(\log n)$ (seemingly unaware of Norton's priority). In the same paper, he considers the situation when $k/\log n \to \infty$, showing that then

$$\limsup_{n\to\infty} \frac{\log d_k(n)}{\log(k/\log n)\cdot \frac{\log n}{\log 2}} = 1. \tag{3}$$

Fedorov says that his proofs involve several cases, and in [**Fed13b**] he restricts attention to when $k = (\log n)^{1+o(1)}$. (But see also [**Fed13a**], which gives detailed arguments when $\frac{\log k}{\log\log n}$ tends to 0 or ∞.) In the survey paper [**CF15**], Chubarikov and Fedorov also claim a sharp result when $k/\log n \to \kappa$ for a fixed $\kappa \in]0,\infty[$ (see Theorem 3.5 on p. 34 there). They assert that a proof can be found in [**Fed13a**], but that paper does not seem to contain the stated theorem or its proof. Moreover, the result itself is incorrect.

The main goal of this paper is to prove a sharp upper estimate for $\log d_k(n)$ in the regime $k/\log n \to \kappa$, thus correcting the work of Chubarikov and Fedorov.

Our starting point is the trivial inequality $d_k(n)n^{-s} \le \sum_{m\ge1} d_k(m)m^{-s} = \zeta(s)^k$. Here, as usual, $\zeta(s)$ is the Riemann zeta function. Taking the logarithm and rearranging,

$$\begin{aligned} \log d_k(n) &\le s\log n + k\log\zeta(s) \\ &= \log n(s+\kappa\log\zeta(s)), \qquad \text{where} \quad \kappa := \frac{k}{\log n}. \end{aligned} \tag{4}$$

For each $\kappa > 0$, the function $s \mapsto s + \kappa\log\zeta(s)$ is continuous on $]1,\infty[$ and diverges to ∞ both as $s \downarrow 1$ and as $s \to \infty$. Thus, it makes sense to define

$$F(\kappa) := \min_{s>1}\left(s+\kappa\log\zeta(s)\right).$$

[1]In fact, it was known to Ramanujan that for fixed k the right-hand side of (2) may be replaced with $\log k \cdot \mathrm{Li}(\log n)$ plus a small error term ([**Ram00**, §39], [**Ram97**, §57]; see also [**Pil44**]). More general results were given by Heppner [**Hep73**] and Nicolas [**Nic80**]. But this interesting line of thought is somewhat orthogonal to the philosophy of this note.

Plugging the minimizing value of s into (4) yields

$$\log d_k(n) \leq F(\kappa) \log n, \tag{5}$$

where as above we write $\kappa = k/\log n$. Since $F(\kappa)$ is easily seen to be continuous on $]0,\infty[$, (5) implies (for instance) that whenever $k/\log n$ tends to a positive limit, we have $\log d_k(n) \leq (F(\kappa)+o(1))\log n$, where $\kappa = \lim \frac{k}{\log n}$.

So far there is little new here. Though (5) is not noted explicitly in Norton's work, its immediate parent (4) appears as [**Nor92**, eq. (5.2)] (see also [**DNR99**, Théorème 1.1]). Our main theorem is that the simple bound (5) is in fact sharp when $k/\log n \to \kappa$ for $\kappa \in]0,\infty[$.[2]

THEOREM 1. *For each fixed $\kappa > 0$, there is a sequence of positive integers k, n with $k/\log n \to \kappa$ such that*

$$\log d_k(n) = (F(\kappa)+o(1))\log n, \qquad \textit{as } n \to \infty.$$

Suppose now that $\kappa > 0$, and that $k, n \to \infty$ with $k \sim \kappa \log n$. In this case, the right-hand side of (2) is asymptotic to $\log n$. But trivially $F(\kappa) > 1$, so that Theorem 1 implies the failure of (2). Hence, Norton's range $k = o(\log n)$ for the validity of (2) is best possible.

We mentioned above Fedorov's result (3) concerning the case when $k/\log n \to \infty$. By analyzing the behavior of $F(\kappa)$ for large κ, we are able to sharpen the upper bound on $\log d_k(n)$ in (3), incorporating a secondary term.

THEOREM 2. *Whenever $\kappa := k/\log n \to \infty$, we have*

$$\log d_k(n) \leq \frac{\log \kappa}{\log 2} \log n + \left(\frac{1+\log\log 2}{\log 2} + o(1)\right) \log n. \tag{6}$$

It is routine — if a bit tedious — to check with Stirling's formula that equality holds in (6) whenever $n = 2^\ell$ and all of k, ℓ and k/ℓ tend to infinity. (One should first recall that $d_k(2^\ell) = \binom{\ell+k-1}{\ell}$.) Thus, (6) is sharp.

REMARK. If one defines $d_k(n)$ as the coefficient of n^{-s} in the Dirichlet series of $\zeta(s)^k$, then the restriction to integral values of k is unnecessary. The entire above discussion remains valid for all real $k \geq 2$. In fact, Norton's results in [**Nor92**] are stated in this more general context.

2. Large values of $d_k(n)$ where $k/\log n \to \kappa$: Proof of Theorem 1

We choose $s \in]1,\infty[$ to minimize $s + \kappa \log \zeta(s)$. By elementary calculus, $-\frac{\zeta'}{\zeta}(s) = \kappa^{-1}$; that is,

$$\sum_{p \text{ prime}} \frac{\log p}{p^s - 1} = \frac{1}{\kappa}.$$

[2]By contrast, Chubarikov and Fedorov claim that $\limsup_{n\to\infty} \frac{\log d_k(n)}{\log n} = \frac{\log(1+\kappa\log(2))}{\log(2)} + \kappa \log\left(1 + \frac{1}{\kappa \log 2}\right)$.

For each prime p, let
$$\beta_p = (p^s-1)^{-1}.$$
With z a parameter at our disposal and $t := z^s$, we let
$$n := \prod_{p\le z} p^{\lfloor \beta_p t\rfloor},$$
so that
$$\begin{aligned}\log n &= \sum_{p\le z} (\beta_p t + O(1))\log p = t\sum_{p\le z}\frac{\log p}{p^s-1} + O(z)\\ &= t\left(-\frac{\zeta'}{\zeta}(s) + O\left(\sum_{p>z}\frac{\log p}{p^s}\right)\right) + O(t^{1/s}) = \frac{t}{\kappa} + O(t^{1/s}).\end{aligned}$$
The final expression is asymptotic to t/κ, as $z\to\infty$. Thus, putting $k := \lfloor t\rfloor$, we see that $k/\log n\to\kappa$.

Moreover,
$$\log d_k(n) = \sum_{p\le z}\log\binom{k+\lfloor\beta_p t\rfloor - 1}{k-1}.$$
Routine (but tedious) calculations with Stirling's formula reveal that
$$\log\binom{k+\lfloor\beta_p t\rfloor-1}{k-1} = t\log(1+\beta_p) + t\beta_p\log(1+1/\beta_p) + O(\log t),$$
and so
$$\begin{aligned}\log d_k(n) &= t\sum_{p\le z}\log(1+\beta_p) + t\sum_{p\le z}\beta_p\log(1+1/\beta_p) + O(t^{1/s})\\ &= t\log\left(\prod_{p\le z}(1-p^{-s})^{-1}\right) + ts\sum_{p\le z}\frac{\log p}{p^s-1} + O(t^{1/s})\\ &= t(\log\zeta(s) + O(z^{1-s})) + ts\left(-\frac{\zeta'}{\zeta}(s) + O(z^{1-s})\right) + O(t^{1/s})\\ &= t\left(\log\zeta(s) + \frac{s}{\kappa}\right) + O(t^{1/s}).\end{aligned}$$
The final right-hand side is asymptotic to $t\kappa^{-1}F(\kappa)$, as $z\to\infty$. Since t is asymptotic to $\kappa\log n$, the theorem follows.

3. When $k/\log n\to\infty$: Proof of Theorem 2

By (5), it suffices to show that as $\kappa\to\infty$,
$$F(\kappa) = \frac{\log\kappa}{\log 2} + \frac{1+\log\log 2}{\log 2} + o(1).$$

Since $\kappa^{-1} \to 0$, the value $s = s_0(\kappa)$ satisfying $-\frac{\zeta'}{\zeta}(s) = \kappa^{-1}$ tends to infinity as $\kappa \to \infty$. In fact, since

$$-\frac{\zeta'}{\zeta}(s) = \sum_{p \text{ prime}} \sum_{\ell \geq 1} \frac{\log p}{p^{\ell s}} \sim \frac{\log 2}{2^s}, \quad \text{as } s \to \infty,$$

we have that $2^{s_0(\kappa)} \sim \kappa \log 2$. Hence,

$$s_0(\kappa) = \frac{\log \kappa}{\log 2} + \frac{\log \log 2}{\log 2} + o(1).$$

Moreover,

$$\log \zeta(s) = \sum_{p \text{ prime}} \sum_{\ell \geq 1} \frac{1}{\ell p^{\ell s}} \sim \frac{1}{2^s} \sim \frac{1}{\log 2}\left(-\frac{\zeta'}{\zeta}(s)\right), \quad \text{as } s \to \infty,$$

and so

$$\begin{aligned} F(\kappa) &= s_0(\kappa) + \kappa \log \zeta(s_0(\kappa)) \\ &= \left(\frac{\log \kappa}{\log 2} + \frac{\log \log 2}{\log 2} + o(1)\right) + (1 + o(1))\kappa \cdot \frac{1}{\kappa \log 2} \\ &= \frac{\log \kappa}{\log 2} + \frac{1 + \log \log 2}{\log 2} + o(1), \end{aligned}$$

as desired.

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