Two thousand years of summing divisors

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Three kinds of natural numbers

> *Among simple even numbers, some are superabundant, others are deficient: these two classes are as two extremes opposed one to the other; as for those that occupy the middle point between the two, they are said to be perfect.*
>
> – Nicomachus (ca. 100 AD), *Introductio Arithmetica*

Let $s(n) = \sum_{d|n, d<n} d$ be the sum of the proper divisors of n.

Abundant: $s(n) > n$, e.g., $n = 12$.
Deficient: $s(n) < n$, e.g., $n = 5$.
Perfect: $s(n) = n$, e.g., $n = 6$.

The superabundant number is . . . as if an adult animal was formed from too many parts or members, having "ten tongues", as the poet says, and ten mouths, or nine lips, and provided with three lines of teeth; or with a hundred arms, or having too many fingers on one of its hands. . . . The deficient number is . . . as if an animal lacked members or natural parts . . . if he does not have a tongue or something like that.

. . . In the case of those that are found between the too much and the too little, that is in equality, is produced virtue, just measure, propriety, beauty and things of that sort — of which the most exemplary form is that type of number which is called perfect.

Is this mathematics?

Abundants: 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, ...

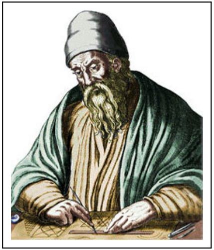
Deficients: 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, ...

Perfects: 6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455991569831744654692615953842176, ...

Questions: Is there a rule to generate the terms in each sequence? Barring that, can we estimate the number of terms up to a given point x?

Nicomachus: No rule needed to generate abundants or deficients.

> *Just as ... ugly and vile things abound, so superabundant and deficient numbers are plentiful and can be found without a rule...*



What about perfect numbers?

Theorem (Euclid)

If $2^n - 1$ is a prime number, then

$$N := 2^{n-1}(2^n - 1)$$

is a perfect number.

Theorem (Euler)

If N is an even perfect number, then N can be written in the form

$$N = 2^{n-1}(2^n - 1),$$

where $2^n - 1$ is a prime number.

But what about odd perfect numbers?
We don't know of a single example.

Dickson's finiteness theorem

Is there a simple formula for odd perfect numbers, like for even perfect numbers? Probably not.

Theorem (Dickson, 1913)

For each positive integer k, there are only finitely many odd perfect numbers n with precisely k distinct prime factors.

Theorem (Pomerance, 1977)

If n is an odd perfect number with k distinct prime factors, then

$$n < (4k)^{(4k)^{2^{k^2}}}.$$

This was refined by Heath-Brown ('94), Cook, and Nielsen:

Theorem

If n is an odd perfect number with k distinct prime factors, then

$$n < 2^{2^{2k}}.$$

Theorem (P., 2010)

The number of odd perfect n with k distinct prime factors is at most

$$2^{(2k)^2}.$$

The web of conditions

> *. . . a prolonged meditation has satisfied me that the existence of [an odd perfect number] - its escape, so to say, from the complex web of conditions which hem it in on all sides - would be little short of a miracle.*
>
> – J. J. Sylvester

If N is an odd perfect number, then:

1. N has the form $p^e M^2$, where $p \equiv e \equiv 1 \pmod 4$,
2. N has at least 9 distinct prime factors and at least 75 prime factors counted with multiplicity,
3. $N > 10^{300}$.

Conjecture

There are no odd perfect numbers.

Densities

If A is a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$, define the *density* of A as

$$\lim_{x \to \infty} \frac{\# A \cap [1, x]}{x}.$$

For example, the even numbers have density $1/2$, and the prime numbers have density 0. But the set of natural numbers with first digit 1 does not have a density.

Question: Does the set of abundant numbers have a density? What about the deficient numbers? The perfect numbers?

A theorem of Davenport

Theorem (Davenport, 1933)

For each real $u \geq 0$, consider the set

$$\mathcal{D}_s(u) = \{n : s(n)/n \leq u\}.$$

This set always possesses an asymptotic density $D_s(u)$. Considered as a function of u, the function D_s is continuous and strictly increasing, with $D_s(0) = 0$ and $D_s(\infty) = 1$.

Corollary

The perfect numbers have density 0, the deficient numbers have density $D_s(1)$, and the abundant numbers have density $1 - D_s(1)$.

Numerics

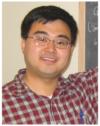
The following theorem improves on earlier work of Behrend, Salié, and Wall:

Theorem (Deléglise, 1998)

For the density of abundant numbers, we have

$$0.2476 < 1 - D_s(1) < 0.2480.$$

So just under 1 *in every* 4 *natural numbers is abundant, and just over* 3 *in* 4 *are deficient.*



Recently Deléglise's results have been improved by Kobayashi.

More on perfect numbers

Let $V(x)$ denote the number of perfect numbers $n \leq x$.
Davenport's theorem says that $V(x)/x \to 0$ as $x \to \infty$.
Can we say anything more precise?

Even perfect numbers correspond to primes of the form $2^n - 1$. We know 47 such values of n, the largest being

$$n = 42643801.$$

Conjecture

The number of $n \leq x$ for which $2^n - 1$ is prime is asymptotic to

$$\frac{e^\gamma}{\log 2} \log x.$$

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The number of perfect numbers up to x is asymptotic to

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Hornfeck & Wirsing, 1957	$V(x) = O(x^{\epsilon})$

The sharpest result to date is due to Wirsing (1959): For all $x > 3$,

$$V(x) \leq x^{W/\log\log x}$$

for a certain absolute constant W. This is no doubt still very far from the truth.

In the opposite direction, the following conjecture is wide open:

Conjecture

There are infinitely many perfect numbers, i.e., $V(x) \to \infty$ as $x \to \infty$.

Almost perfect?

If n is perfect, then $n \mid s(n)$, and so $\gcd(n, s(n))$ is as large as possible.

Theorem (P., 2009)

Fix α with $0 \le \alpha \le 1$. The number of $n \le x$ for which

$$\gcd(n, s(n)) \ge n^{\alpha}$$

is $x^{1-\alpha+o(1)}$ as $x \to \infty$.

The proof uses Wirsing's result and some ideas of Luca and Pomerance used to study the Euler φ-function.

Local distribution

How are deficients, abundants, and perfects distributed in intervals $[x, x+y]$, where y is much smaller than x?

Theorem (I. M. Trivial)

For $n > 6$, the interval $(n, n+6]$ contains an abundant number.

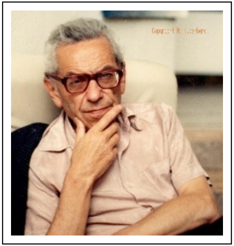
Proof.

If $n = 6k$ and $k > 1$, then $s(n) \geq 1 + k + 2k + 3k = 6k + 1 > n$.

So there is no gap of length > 6 between abundant numbers.
(It can be shown that each gap of size ≤ 6 occurs infinitely often.)

How large can the gap be between consecutive deficient numbers? Alternatively, how long can a run of abundant numbers be?

Answer: Arbitrarily long, by the Chinese remainder theorem. But we can be more precise:



Theorem (Erdős, 1934)

Let $G(x)$ be the largest gap $n' - n$ between two consecutive deficient numbers $n < n' \leq x$. There are constants positive constants c_1 and c_2 with

$$c_1 \log\log\log x < G(x) < c_2 \log\log\log x.$$

Theorem (P., 2009)

Let $G(x)$ be the largest gap $n' - n$ between two consecutive deficient numbers $n < n' \le x$. As $x \to \infty$, we have

$$\frac{G(x)}{\log\log\log x} \to C,$$

where $C \approx 3.5$. In fact,

$$C = \left(\int_0^1 \frac{D_s(u)}{u+1}\, du\right)^{-1}.$$

The local distribution of perfect numbers seems much harder to understand rigorously.

Higher order generalizations

Problem (Catalan, 1888)

Start with a natural number n. What is the eventual behavior of the sequence of iterates $n, s(n), s(s(n)), s(s(s(n))), \ldots$ (the *aliquot sequence at n*)?

Example

$n = 20$ leads to the sequence $20, 22, 14, 10, 8, 7, 1, 0$.

Example

$n = 25$ leads to the sequence $25, 6, 6, 6, 6, 6, \ldots$.

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Every starting n leads to a sequence terminating at 0, or hits a perfect number.

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Example

$n = 220$ leads to the sequence $220, 284, 220, 284, 220, 284, \ldots$.

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Example

$n = 220$ leads to the sequence $220, 284, 220, 284, 220, 284, \ldots$.

Conjecture (Catalan–Dickson, 1913)

Every starting n leads to a bounded sequence, i.e., either a sequence terminating in 0 or reaching a cycle.

This conjecture is true for every $n < 276$, but no one knows if it holds for $n = 276$.

Definition

We call n a *sociable number* if the aliquot sequence at n is purely periodic; in this case, the length of the period is called the *order* of sociability.

Definition

An *amicable number* is a sociable number of order 2. In this case, the pair $\{n, s(n)\}$ is an amicable pair.

Ibn Khaldun (ca. 600 years ago):

> *Persons who have concerned themselves with talismans affirm that the amicable numbers 220 and 284 have an influence to establish a union or close friendship between two individuals.*

Abraham Azulai (ca. 500 years ago):

> *Our ancestor Jacob prepared his present in a wise way. This number 220 is a hidden secret, being one of a pair of numbers such that the parts of it are equal to the other one 284, and conversely. And Jacob had this in mind; this has been tried by the ancients in securing the love of kings and dignitaries.*

Al-Majriti (ca. 1050 years) claims to have tested the erotic effect of

> *giving any one the smaller number 220 to eat, and himself eating the larger number 284.*

The (global) distribution of sociable numbers

Let $V_k(x)$ denote the number of sociable numbers of order k not exceeding x. (So $V(x) = V_1(x)$.)

Conjecture (Bratley, Lunnon, and McKay)

$V_2(x)/x^{1/2} \to 0$ as $x \to \infty$.

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Though we know > 12 million amicable pairs, the following is still open:

Conjecture

There are infinitely many amicable numbers.

Theorem (Erdős 1955, Erdős and Rieger 1975)

The set of amicable numbers has density zero. In fact,

$$V_2(x) \ll \frac{x}{\log\log\log x}.$$

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Theorem (Pomerance, 1981)

For all large x,

$$V_2(x) \le \frac{x}{\exp((\log x)^{1/3})}.$$

As a consequence, the sum of the reciprocals of the amicable numbers converges.

Sociable numbers of order > 2 have a more recent pedigree.

Example

Here is a sociable cycle of order 5 (found by Poulet in 1918):

$$12496 \to 14288 \to 15472 \to 14536 \to 14264 \to 12496 \to \ldots$$

order of the cycle	number of known examples
1	47
2	$>$ 12 million
4	165
5	1
6	5
8	2
9	1
28	1

Theorem (Erdős, 1976)

For each fixed k, the set of sociable numbers of order k has density zero.

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where the denominator is a $(3k)$-fold iterated log.

With Kobayashi and Pomerance, we improved this. A recent result is:

Theorem (P., 2009)

For each fixed odd value of k,

$$V_k(x) \leq \frac{x}{(\log x)^{1+o(1)}}.$$

All together now

Put

$$V^*(x) = V_1(x) + V_2(x) + V_3(x) + \dots,$$

so that $V^*(x)$ is the counting function of *all* the sociable numbers.

Conjecture

As $x \to \infty$, we have $V^*(x)/x \to 0$. In other words, the set of sociable numbers has density zero.

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Theorem (Kobayashi, Pomerance, P., 2009)

The set of deficient sociable numbers has density zero. The set of even abundant sociable numbers has density zero. Finally, the set of odd abundant numbers has density $\approx 1/500$.

Thank you for your attention!